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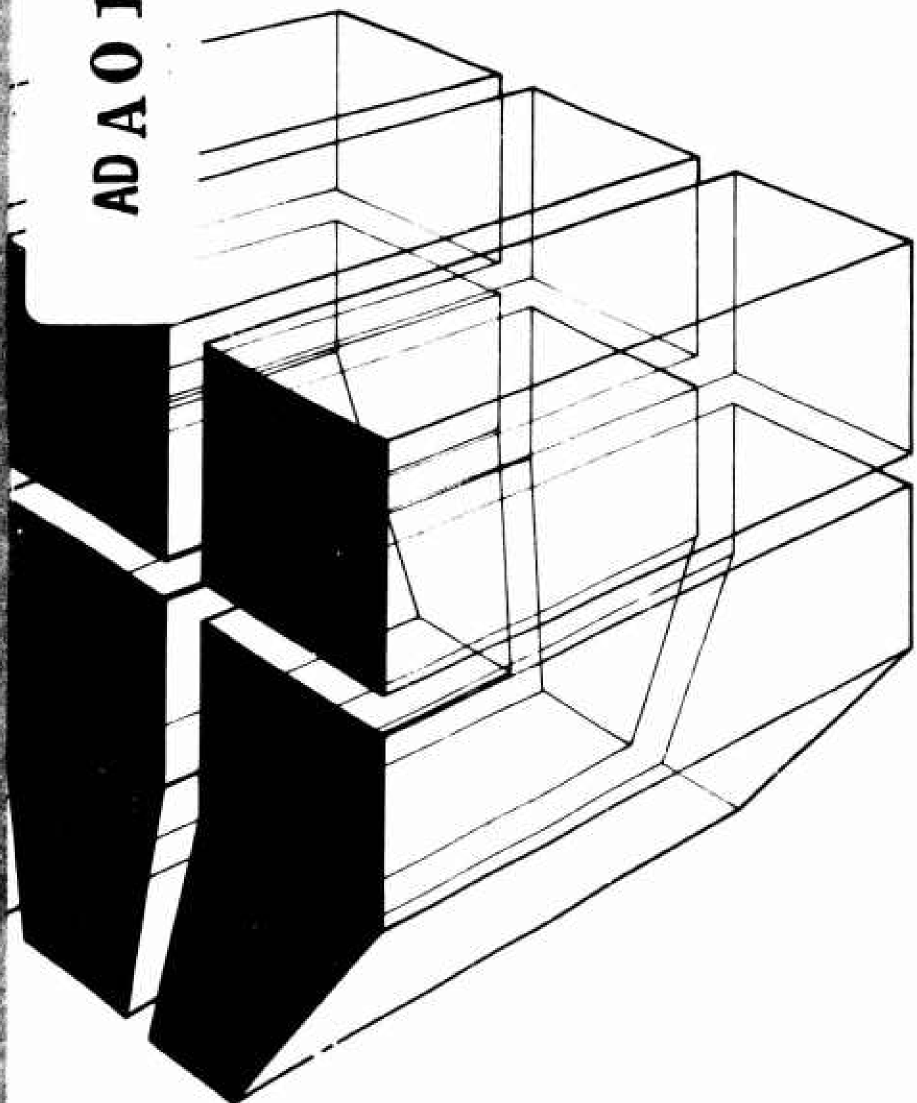
June 1975
Earthquake Effects on Structures

MODAL ANALYSIS METHODS IN SEISMIC DESIGN FOR BUILDINGS



by
William K. Stockdale

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This report is the first step in preparing a change to the tri-services manual TM 5-809-10, *Seismic Design for Buildings*. Changes in this manual are necessary to provide guidance for the design of critical military facilities which must remain functional after subjection to strong earthquakes. This report describes and discusses modal analysis methods used in the dynamic analysis of structures in conjunction with the earthquake response spectra and time history methods. Elastic and inelastic conditions are discussed, as well as structural damping and assumptions and limitations of the methods. Example calculations are included.

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FOREWORD

This project was conducted for the Directorate of Military Construction, Office of the Chief of Engineers (OCE), under RDT&E Army Program 6.27.19A, Project 4A76719AT05, "Initial Investigation in Military Construction Technology," Task 02, "Engineering Design Criteria and Technology for Military Facilities," Work Unit 004, "Earthquake Effects on Structures." The OCE Technical Monitor was Mr. W. A. Heitmann.

This study was conducted between August 1974 and February 1975 by the Structural Mechanics Branch, Materials Systems and Science Division (MS), Construction Engineering Research Laboratory (CERL). This report was prepared by COL W. K. Stockdale, Department of Engineering, U.S. Military Academy, West Point, NY, while on sabbatical at the University of Illinois, Urbana-Champaign. The work was accomplished under the supervision of Dr. W. E. Fisher, Chief, Structural Mechanics Branch.

COL M. D. Remus is Commander and Director of CERL, and Dr. L. R. Shaffer is Deputy Director. Mr. J. J. Healy is Chief of MS.

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MODAL ANALYSIS METHODS IN SEISMIC DESIGN FOR BUILDINGS

1 INTRODUCTION

Purpose

The purposes of this report are (1) to summarize modal analysis techniques that can be used to assist in the analysis of the dynamic response of buildings subjected to seismic motions, and (2) to develop preliminary recommendations for implementing modal analysis techniques to supplement the equivalent static force method specified in the current version of TM 5-809-10.[1]

Background

The seismic provisions of the Structural Engineers Association of California (SEAOC) Code adopted in TM 5-809-10 are designed to insure incorporation of acceptable minimum strengths into a structure. These minima have been established by a consensus of engineers and building department officials who have been guided by observations and calculations regarding the performance of various types of structures.

Deficiencies in the code are revealed by unsatisfactory performance of structures in past earthquakes, and improvements are brought about by experience. For most aspects of engineering design, experience is a rapid teacher and feedback is quick. However, the relatively infrequent occurrence of strong and destructive earthquakes means that experience is relatively slow in focusing attention on deficiencies in seismic design and construction. When experience does occur, it may be too late to prevent significant loss of life and destruction of property.

The seismic design provisions specified in the SEAOC code are based primarily on the first mode response of the structure; they substitute a set of equivalent static lateral forces for the true dynamic forces imposed on the structure by the seismic motion (Figure 1(a)). The basic concept of the SEAOC code is contained in the formula for the equivalent base shear (V) given by

$$V = ZKCW \qquad \text{[Eq 1]}$$

where

- Z = a coefficient dependent upon the relative intensity of the ground motion at the site of the structure
- K = a coefficient recognizing the effect of ductility and energy absorption qualities of certain types of construction which have historically shown varying degrees of earthquake resistance
- C = a coefficient recognizing the effect of the period of the structure on the response to the ground motions
- W = the total weight of the structure.

Furthermore, the SEAOC code distributes the base shear vertically along the height of the structure in a linear manner that approximates the inertial loading imposed on the structure when it responds in its fundamental mode of vibration. The largest force is applied at the top of the structure, with the force decreasing to zero at the base (Figure 1(a)). This is accomplished by use of the formula

$$F_x = \frac{(V - F_t)w_x h_x}{\sum_{i=1}^{n} w_i h_i} \qquad \text{[Eq 2]}$$

where F_x = force applied at any floor x (including top)

F_t = an extra force applied to the top of the structure = $.004V\left(\frac{h_n}{D_s}\right)^2$; $= 0$ if $\frac{h_n}{D_s} \leq 3$

h_n = height in feet to top of structure

D_s = dimension in feet of structure at base in direction being analyzed

h_i = height in feet to any floor i (including top)

w_i = weight in kips of any floor i (including top).

From an economic point of view, the amount of investment that should be made in the seismic design of a structure is limited; that is, there is an optimum point beyond which the extra cost of the design effort

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[1] *Seismic Design for Buildings*, TM 5-809-10/NAVFAC P-355/AFM 88-3, Chapter 13 (Departments of the Army, Navy, and Air Force, April 1973).

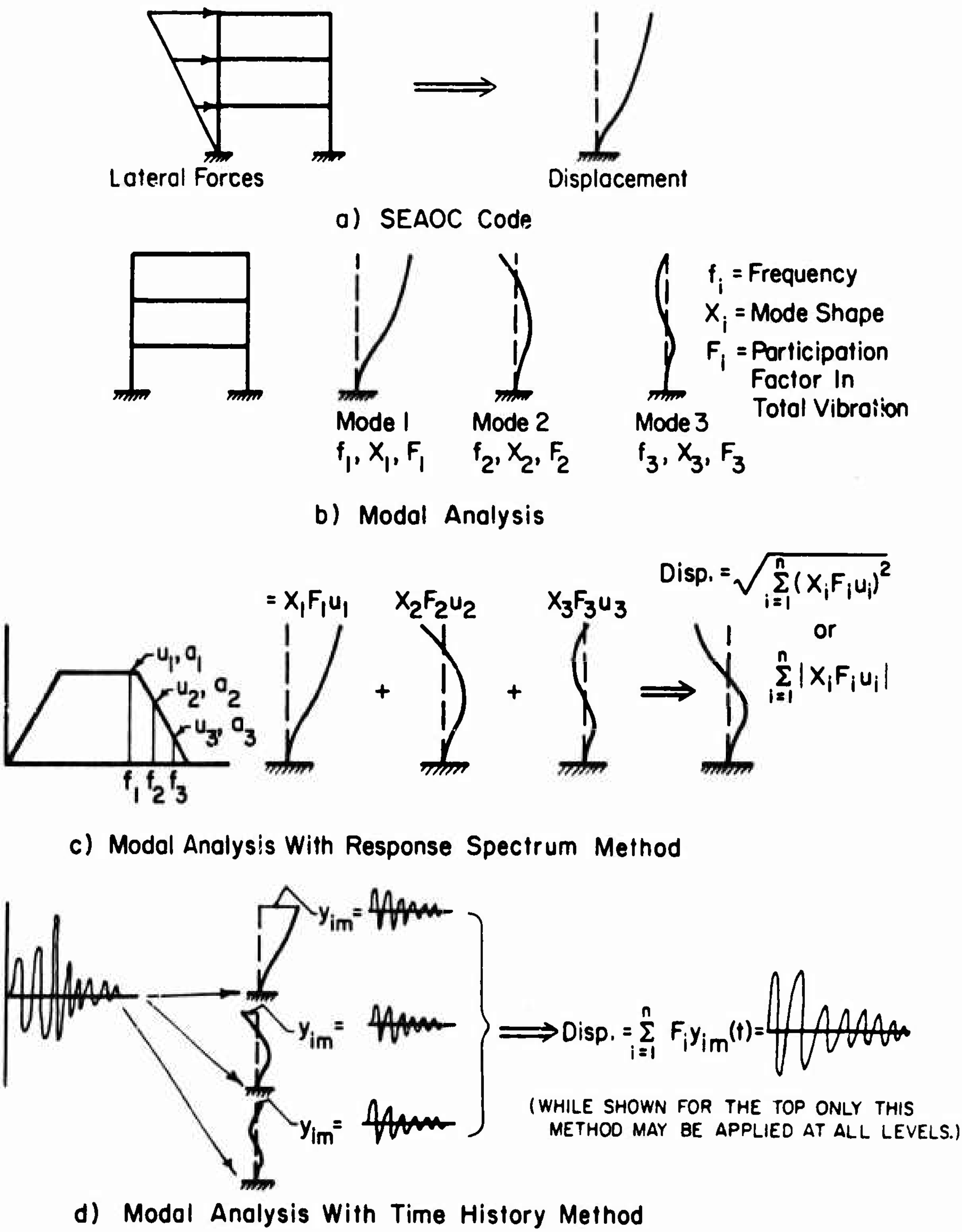

Figure 1. Comparison of SEAOC code and modal analysis methods of seismic design.

exceeds the resulting increase in the structure's value or the reduction in construction costs. The equivalent static force method, with some adjustment in the various coefficients comprising the method, approaches the optimum method for the majority of conventional, non-critical, low-rise construction. However, for critical military facilities which are required to support relief and rescue operations and retain military defense integrity immediately after an earthquake, the use of equivalent static force methods is not always adequate to insure survivability. For critical facilities, more reliable and refined methods of design must be implemented to eliminate the uncertainty associated with the current code.

Primarily as a result of the emphasis on safe seismic design of nuclear reactors and the 1971 San Fernando earthquake, various improved procedures for the seismic design of facilities have recently been proposed, and some have been implemented. Most notable have been the design standards adopted by the Nuclear Regulatory Commission (NRC, formerly AEC). These standards have tended to specify the seismic hazard in terms of a basic design spectrum, which is commonly normalized to a maximum horizontal ground acceleration of 1.0 g, but is capable of being scaled to other acceleration levels to satisfy site conditions (Figure 2). Thus, these standards encourage using response spectrum modal analysis techniques for analyzing the dynamic response of a proposed or existing structure. The alternate approach is to use a family of actual past or artificial earthquake records scaled to specific parameters established during a site-dependent investigation to perform a time history modal analysis to compute the transient response of the structure.

Response spectrum and time history modal analysis methods are not new, but they have been made practical with the availability of large capacity computers and a number of general purpose com-

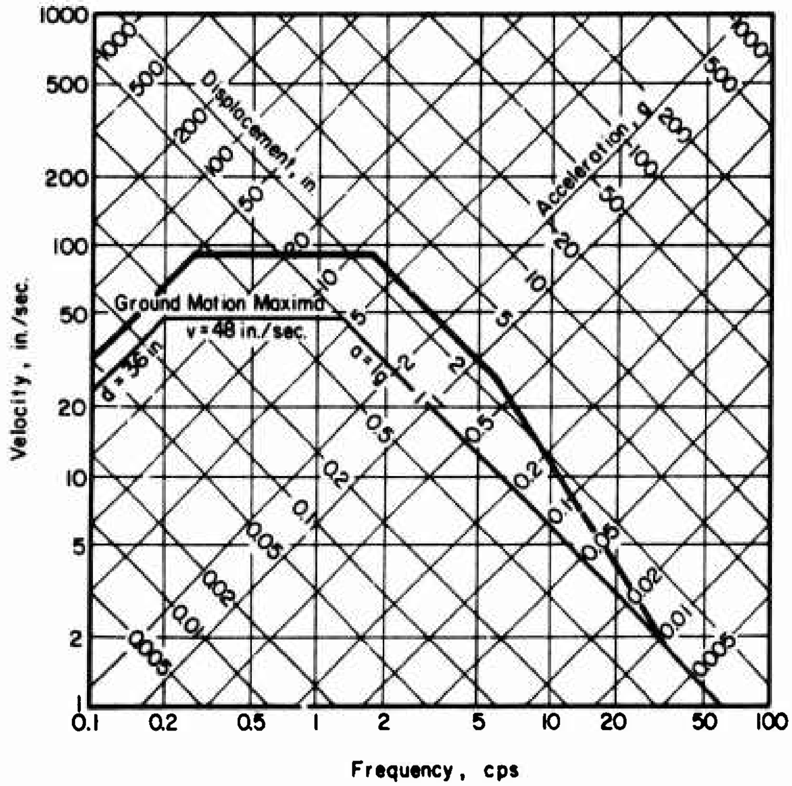

Figure 2. Basic design spectrum normalized to 1.0 g for 5 percent damping. From N. M. Newmark and W. J. Hall, "Procedures and Criteria for Earthquake Resistant Design," *Building Practices for Disaster Mitigation* (Department of Commerce, February 1973).

puter programs[2] developed for dynamic analysis of structures (Figures 1(b, c, and d)). Both methods provide better understanding of the nature of the dynamic response of a structure and a more practical method of analysis and design. Other time history methods may be used for analysis of more complicated structures than those considered in this report.

2 MODAL ANALYSIS METHODS

Description of Modal Analysis Method

All structures and their contents are complicated systems whose dynamic characteristics are not amenable to exact evaluation. Similarly, earthquake tremors are random motions whose direction and magnitude are not accurately predictable. Thus an exact analysis of how an existing structure will respond to future earthquakes or if a designed new structure can successfully withstand all future earthquakes is difficult if not impossible. To approach these analysis and design processes rationally, simplifying assumptions and engineering judgment are essential. The method currently used most often for critical facilities is the modal analysis method.

The terminology "modal analysis method" comes from the concept of separating a vibration system into its principal modes. All vibrating systems consist of a vibrating mass or masses and elements which tend to resist the motion or displacement of the mass or masses. The resisting elements are usually idealized as (1) either elastic or inelastic springs which tend to make the mass or masses return to a minimum displacement position, and (2) some energy-absorbing or frictional system which tends to dampen the motion of the masses. Some simple typical vibrating systems are shown in Figure 3.

Assuming the spring stiffness (k) of the system represented by Figure 3(a) is linear, the motion of the mass will be sinusoidal with respect to time when the system undergoes free vibrations. Since the system has only one mass and one spring in the horizontal direction, it has only one degree of freedom, or one typical motion; thus, it has one mode. Its motion can be represented by the equation

$$X(t) = X_{max} \sin \omega t \qquad \text{[Eq 3]}$$

where X = the displacement of the mass relative to the base at any time t

X_{max} = the maximum displacement relative to the base

ω = the natural frequency of the system in radians per unit of time.

The system represented in Figure 3(b), however, has three masses, each of which moves according to the forces acting on it at any instant of time. While these masses are not independent, neither do they necessarily move as a unit. In fact, they would typically assume three different characteristic mode

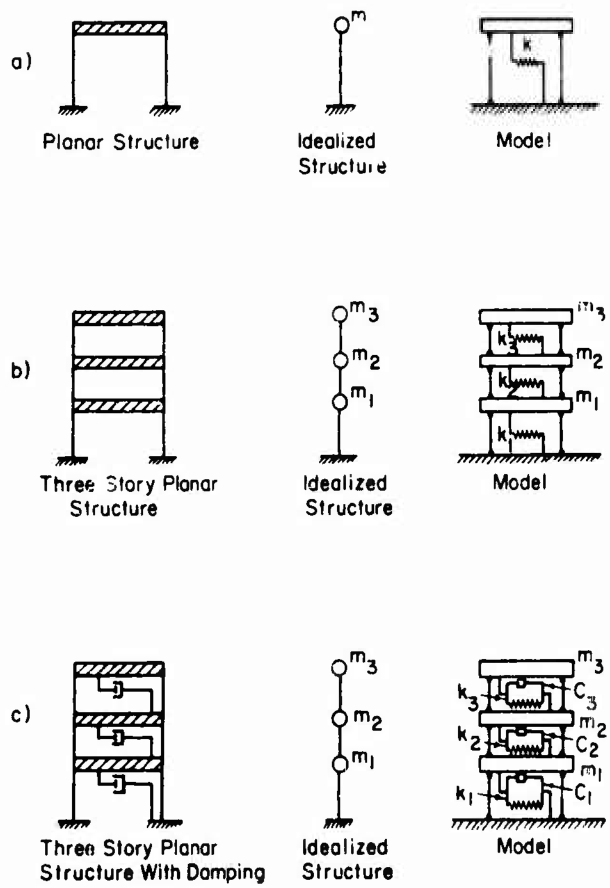

Figure 3. Typical vibrating systems.

[2] Agbabian-Jacobsen Associates, *User's Guide for GENSAP Code* (U.S. Army Corps of Engineers, Huntsville Division, May 1972); and E. L. Wilson and H. H. Dovey, *Three Dimensional Analysis of Building Systems—TABS*, Report No. EERC 72-8 (University of California, December 1972).

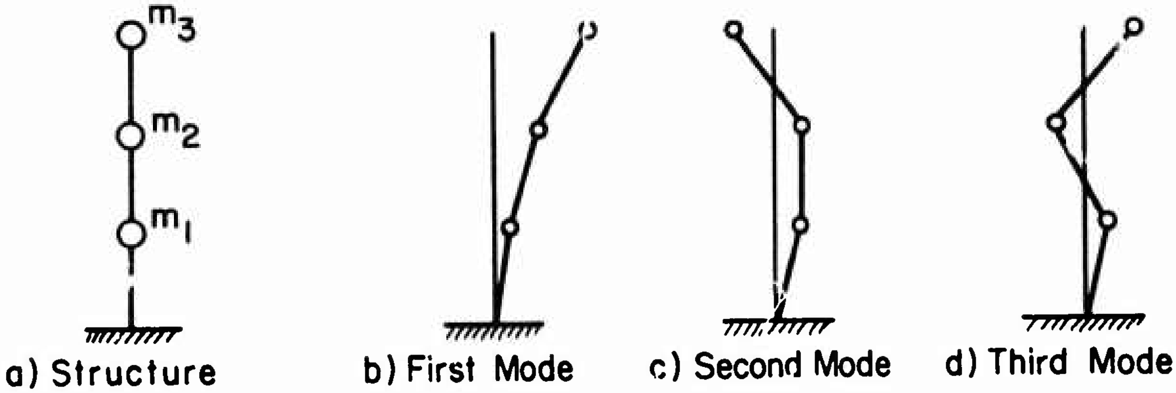

Figure 4. Three principal mode shapes of system represented in Figure 3(b).

shapes (Figure 4). Each of these principal mode shapes would have a unique frequency of vibration (ω) and each mass would have a particular displacement for each mode shape (X_{nm}). The displacement represented by Figure 4(b) is typically referred to as the first or fundamental mode, while (c) and (d) represent the second and third principal modes, respectively. The exact values for X_{nm} for each mass are functions of the masses themselves and the spring stiffnesses.

If the system is vibrating freely in one of its principal mode shapes, each mass will follow Eq 3 or

$$X_{nm}(t) = (X_{nm})_{max} \sin \omega_n t \qquad \text{[Eq 4]}$$

If the system is disturbed by forces applied to the masses themselves or by base disturbance (such as an earthquake motion), each of these mode shapes will be excited to some extent. The amount that each mode shape contributes to the total response is known as its participation in the motion. The total free vibration of each mass relative to the base is represented by

$$X_m(t) = \sum_{n=1}^{n} F_n (X_{nm})_{max} \sin \omega_n t \qquad \text{[Eq 5]}$$

where F_n represents the participation of mode n in the total response of the system. The forced vibration response is discussed under "Time History Method."

The use of Eq 5 in the analysis of dynamic systems is generally referred to as the modal analysis method. While the equation represents deflection only, similar relationships exist for velocities, accelerations, shears, moments, and other linear functions of interest. Appendix A contains a more complete discussion of modal analysis.

Response Spectrum Method

When a system is subjected to a forcing function applied to the system base, such as an earthquake motion, its particular response is a function of the system characteristics and the properties of the forcing function. A typical example of the displacement response spectrum for single-degree-of-freedom systems subjected to an earthquake motion is shown in Figure 5. In this case, the natural frequency of the structure, f($f = \omega/2\pi$) is plotted along the horizontal axis, and the maximum displacement of the mass (X_{max}) is plotted along the vertical axis.

The displacement, velocity, and acceleration are related in the following way:

$$\text{Displacement} = X(t) = X_{max} \sin \omega t \qquad \text{[Eq 6]}$$

$$\text{Velocity} = \dot{X}(t) = \omega (X_{max}) \cos \omega t \qquad \text{[Eq 7]}$$

$$\text{Acceleration} = \dot{V} = \ddot{X}(t) = -\omega^2(X_{max}) \sin \omega t \qquad \text{[Eq 8]}$$

where $\dot{}$ represents a derivative with respect to time. The maximum absolute values for the displacement, velocity, and acceleration relative to the base are X_{max}, ωX_{max}, and $\omega^2 X_{max}$, respectively. Consequently, if the natural frequency and either the

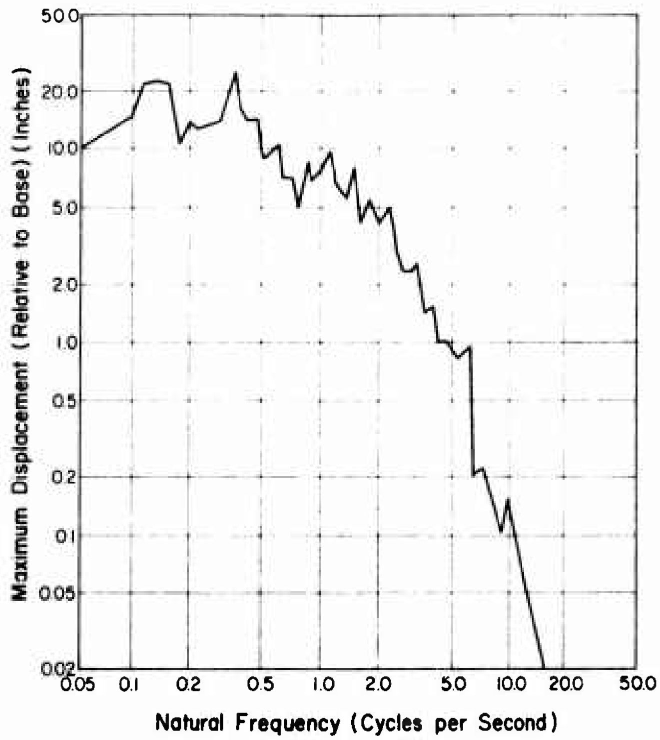

Figure 5. Displacement response spectrum for single-degree-of-freedom systems subjected to earthquake motion. Constructed from N. M. Newmark and W. J. Hall, "Procedures and Criteria for Earthquake Resistant Design," *Building Practices for Disaster Mitigation* (Department of Commerce, February 1973).

maximum displacement, velocity, or acceleration of a system for a particular forcing function are known, the other two maxima are also known. A typical response spectrum used in earthquake analysis where all three maxima are of interest is shown in Figure 6, a replotting of the response shown in Figure 5.

For linear systems, it can be theoretically shown that the response of two or more systems having the same frequency (ω) will be the same for the same forcing function. This fundamental concept of modal analysis allows prediction of the maximum behavior of large systems through studying the response of simple systems. The frequency is the link between the two when they are subjected to the same motion.

A complication in the application of this method to large systems occurs because large systems usually have several principal mode shapes, each with its own ω value. Since these mode shapes will generally not be in phase with each other, their maximum values will not occur at the same time. Thus, simply adding the maximum values for the various modes to obtain the maximum for the system is not possible. Instead, they must be combined in such a way that the expected response of the total system results. Three general methods are used: (1) the fundamental mode alone, since it is usually dominant in the response; (2) the absolute maxima of the several mode shapes, to achieve a conservative upper bound of the response; or (3) a square root of the sum of the squares approach, which takes account of the probability that the maxima of the modes do not all occur at the same time (Figure 1(c)). The problem is discussed more completely under "Basic Procedures."

A further complication is the accuracy of the response spectrum itself in predicting future motions. Each spectrum prepared to date has been based upon measurements taken at a particular point for a particular earthquake. One cannot now predict with reasonable assurance of being accurate what motions will occur at any particular geographical

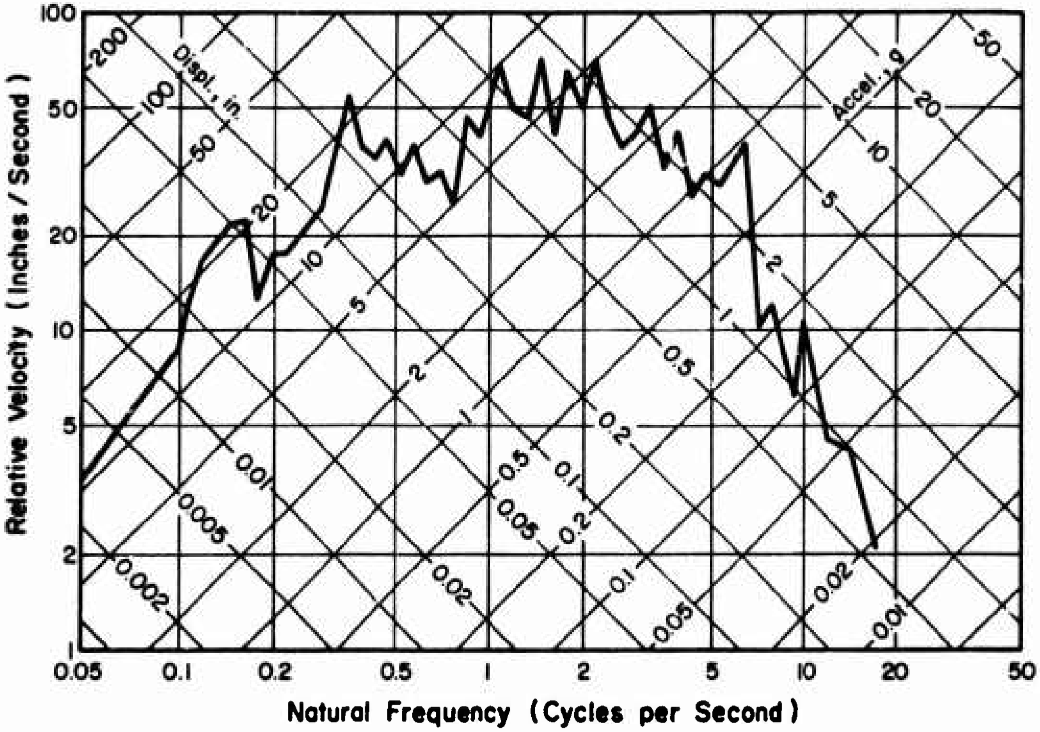

Figure 6. Typical response spectrum. From N. M. Newmark and W. J. Hall, "Procedures and Criteria for Earthquake Resistant Design," *Building Practices for Disaster Mitigation* (Department of Commerce, February 1973).

point of interest. Additionally, experience indicates that even at the same geographical location the ground is subjected to motions of various intensities during various earthquakes; this is reinforced by the recognition that earthquake sources vary in locations. Thus no reliable way of predicting what motions a particular system will experience in its lifetime exists.

However, the data accumulated so far have shown that there are typical patterns of response spectra which result from earthquake motions. In fact, bounds on the maximum expected behavior can be drawn as shown in Figure 7. Additionally, as more data are collected and microseismic techniques perfected, it is expected that for any particular site, response spectra bounds will be related to the probability of occurrence of those motions. This relationship will be similar to the way the runoff in a particular stream can be related to the probability of occurrence of a storm of particular intensity (a 100-year storm, a 10-year storm, etc.).

Use of the response spectrum method assumes that an appropriate response spectrum for a particular site is available.

Time History Method

Use of the response spectrum with the modal analysis technique described above yields only the maximum values of the functions such as displacement, acceleration, shear, or moment. In some cases, how a function varies with time during an earthquake may also be of interest.

Again using the modal analysis method in conjunction with an appropriate earthquake acceleration record, the response of several single-degree-of-freedom systems with frequencies similar to the frequencies of the real system being analyzed are computed. Using a time history representation, these computations are made in a series of time steps, starting with initial conditions and taking a small time interval and computing the response at the end

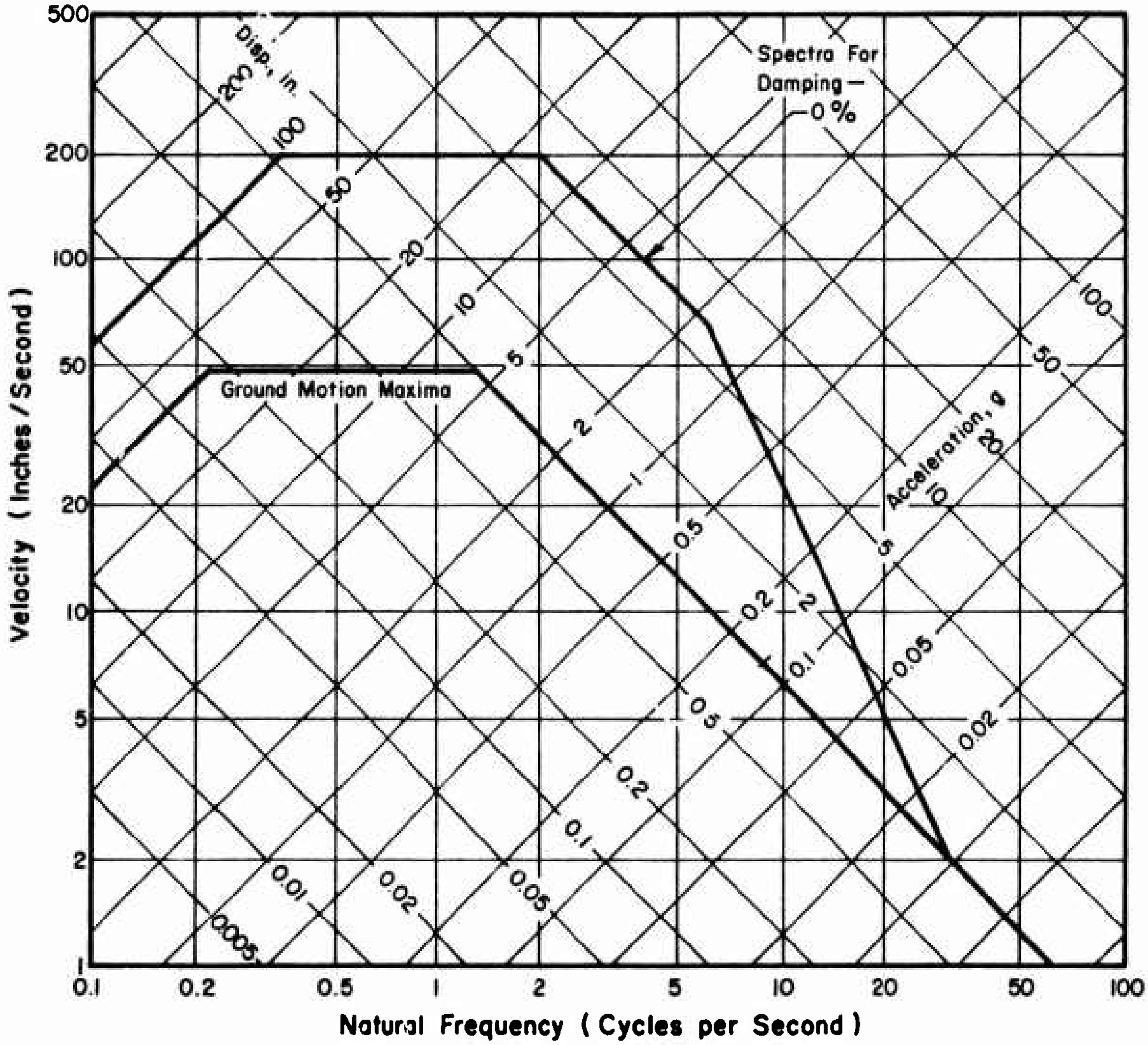

Figure 7. Basic spectrum normalized to 1.0 g maximum acceleration. From N. M. Newmark and W. J. Hall, "Procedures and Criteria for Earthquake Resistant Design," *Building Practices for Disaster Mitigation* (Department of Commerce, February 1973).

of that Δt. The responses of the mode shapes of the real system are then computed and combined using appropriate participation factors to obtain the response of the real system as a function of the time (Figure 1(d)). In equation form this is:

$$x_m(t) = \sum_{n=1}^{n} F_n X_{nm} \mu_n(t) \qquad \text{[Eq 9]}$$

where μ_n is the response of the corresponding single-degree-of-freedom system to the earthquake at the end of that Δt interval, and is a function of the period or frequency of the system and the particular earthquake motion input.

The response is then computed using another time interval. This operation continues until the maximum conditions are encountered or the end of the time period of interest is reached.

The calculations are best done by computer since they are long and repetitious.

In most cases this method will give more exact values for the maximum velocities, displacements,

shears, etc. than the response spectrum method. However, the values calculated apply only to one particular acceleration record. Thus while they may approximate the motions for the prior earthquake, they may have no meaning for the next. Consequently, the analyst has to balance the increased cost of the computer time and the accuracy of the results achieved, and decide whether to use the response spectrum method or the time history method. Additionally, while the time history method appears to yield more accurate results, there are several possibilities for error, and results may not be as accurate as they first appear.

Assumptions and Limitations

When simplifying anything as complicated as the response of a structure to earthquakes, several assumptions and some limitations to the use of the analysis methods are required. Some of these have already been discussed.

Input Motions

Some discussion of the input motions is in the previous section. When analyzing a particular building for a particular site, knowing what motions can be expected when would be helpful. This is currently impossible with any degree of accuracy. After detailed study of the earthquake experience and the soil and rock conditions of an area, experts can predict with some degree of assurance a band of motions which might be expected and how often they might occur; however, an earthquake of a particular intensity might occur tomorrow, or it might occur 100 years from now. Thus, when particular ground motions are selected for investigation and use in the design process, a certain level of protection which may or may not be exceeded in the structure's anticipated life span is provided.

Linear Elastic Assumptions

The modal analysis method is based on the assumption that the structure remains elastic, or nearly so, during the entire earthquake. This means that there is either very little or no permanent deformation in the structure. For small or moderate earthquakes, this is not a bad assumption for well-designed structures. However, for strong or very strong motions, this may be a very poor assumption even for well-designed structures.

While most effort to date has been spent on linear elastic analysis, considerable effort is now underway to extend the design procedures into the inelastic range.

The linear elastic assumptions lead to two major problem areas:

(1) Concrete, one of the most important structural materials, is elastic over only a small range of strain. Distortion quickly results in a cracked section; this significantly changes its load-deflection relationships. Additionally, its effective modulus of elasticity changes drastically, and the concrete crushes (loses all strength) if strained much at all. These effects can be considered in analysis, but seriously complicate the modal analysis method described above.

(2) Inelastic action seriously affects the period of vibration and may cause the structure to undergo an entirely different kind of response than that predicted by the elastic analysis methods described above. In general, inelastic action extends the period of the structure; that is, it takes longer to make one complete cycle of vibration. This might change the response of the structure from one in which acceleration is critical to one in which deflection is critical.

While these problems can be dealt with, they do seriously complicate the linear elastic analysis.

Damping in the Structure

Every vibrating system, including structures subjected to earthquake motion, loses energy in some way. If it did not, it would continue vibrating forever once it is started. This loss of energy in structural systems is called damping. Damping can be of several types, the most common of which are frictional and viscous damping. Frictional or coulomb damping is the kind that occurs when a chair is pushed across a floor. The total energy used is a function of the force pressing the two surfaces together, the coefficient of friction, and the distance moved.

Viscous damping is a function of the velocity of the mass and the characteristics of the system. Since most of the energy absorbed in a system is internal to the structural materials themselves, this kind of damping best represents the energy loss of the

system and is used most often in this type of analysis.

A difficult step in modeling a structure is determining the amount of damping to use to represent the energy loss. The amount of damping used is usually specified in terms of critical damping, which is the amount of damping that prevents oscillating motion. The effects of subcritical and supercritical damping for a single-degree-of-freedom system are shown in Figures A4 and A5, respectively. Usually, the amount of damping exhibited in real structures is between 0 and 10 percent of critical damping, depending upon the type of materials used and the structural concept employed. Some currently recommended values are shown in Table 1.

Table 1

Recommended Damping Values*

Stress Level	Type and Condition of Structure	Percentage of Critical Damping
Working stress, no more than about one-half yield point	Vital piping	0.5 to 1.0
	Welded steel, prestressed concrete, well reinforced concrete (only slight cracking)	2
	Reinforced concrete with considerable cracking	3 to 5
	Bolted and/or riveted steel, wood structures with nailed or bolted joints	5 to 7
At or just below yield point	Vital piping	2
	Welded steel, prestressed concrete (without complete loss in prestress)	5
	Prestressed concrete with no prestress left	7
	Reinforced concrete	7 to 10
	Bolted and/or riveted steel, wood structures, with bolted joints	10 to 15
	Wood structures with nailed joints	15 to 20

*N. M. Newmark and W. J. Hall, "Procedures and Criteria for Earthquake Resistant Design," *Building Practices for Disaster Mitigation* (Department of Commerce, February 1973).

When systems with more than one degree of freedom are analyzed, the damping acts between floors (masses) as represented by the dashpots and masses in Figure 3(c). Thus, the damping depends upon the relative velocities between the masses or between the first mass and the foundation.

To make modal analysis possible, the amount of damping between masses must be consistent with the other structural characteristics, the masses, and the spring stiffnesses.

Appendix A contains a more complete discussion of damping.

Basic Procedures

The procedures discussed below provide for the analysis of typical buildings such as hospitals, warehouses, offices, and billets. The analysis of more complex structures such as dams, nuclear reactors, and nonlinear structures requires more detailed evaluation than is provided by the following procedures.

Analysis of structures subjected to earthquake motions requires several simplifying assumptions and considerable engineering judgment. In addition, of course, it requires many calculations. While these can be performed by hand, they are most often done by a digital computer with programs written specifically for this type of analysis.

When using these prepared programs, however, the analyst should be careful to understand the method of solution, the modeling techniques, and the assumptions employed in preparing the program. Additionally, he should recognize that the answers are accurate to only one or perhaps two significant figures.

The procedures described below represent a general approach to analysis. With some structures or some prepared programs, other procedures or steps may be required.

It is assumed at this point that a structure has already been designed on a preliminary basis. A structural concept must be formed with certain members and equipment determined or a fairly accurate estimate of their mass and strength made. Since design is an iterative process, the results of any one dynamic analysis may require modifications in the design of a structure under consideration. How-

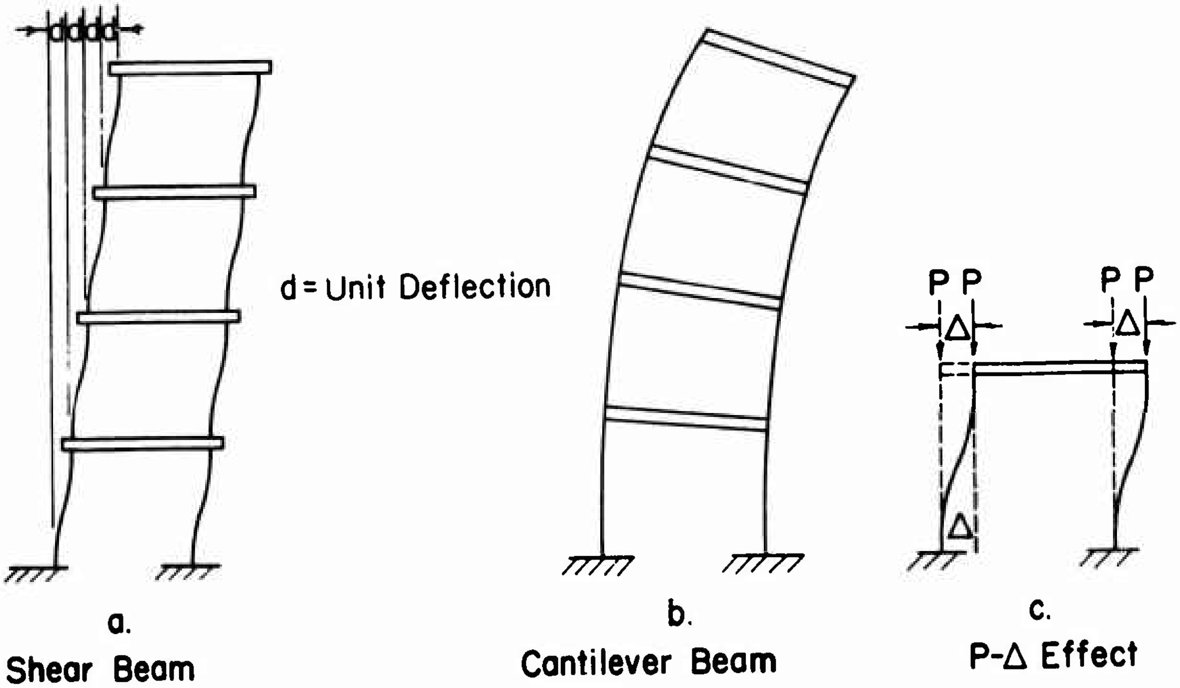

Figure 8. Types of structural response.

ever, certain decisions must already have been made, at least tentatively, before this analysis can be performed.

Idealization of Structural System and Calculation of Member Properties and Masses

As a first step in analyzing the response of a structure to earthquake motion, the engineer must model the structure. An overall evaluation of the structure is required to define the vertical and lateral force resistant system, and to determine where the stiffnesses, masses, and energy-absorbing systems of the structure are, and how they are linked together. Assuming Figure 3(c) is a typical representation, the engineer then determines the values of k, c, and m for the stiffness, damping, and mass respectively, of the various parts of the structure.

It is usually not too difficult to evaluate the mass of a typical structure. All dead load that is expected, all live load that can reasonably be expected to be attached to the structure, and in some cases, such as a warehouse, some percentage of the design live load must be included. Normally, the rest of the design live load is considered to move around within a structure and does not contribute to the horizontal response of the structure.

For modeling purposes, the masses must be lumped at discrete points. Usually the mass of walls, partitions, and vertical structural elements are consolidated at the floor level. For these portions of the structure, half the mass goes to the floor system and half goes to the next higher floor (or roof) system.

Determining the stiffness of the individual members and the structure as a whole is a more complicated problem. Some structures will respond like a shear beam (Figure 8(a)), while others will respond like a cantilever beam (Figure 8(b)). In some cases both responses must be considered.

If a shear beam analysis is appropriate, the analyst must calculate the force it takes to deform the members on the basis of a unit deflection (Figure 8(a)). Summing all these for one floor in one direction gives the stiffness against motion (k) for that mass in that direction. A similar analysis for any other direction of interest and all other masses must also be performed.

If a cantilever beam analysis is appropriate, the analyst must calculate the resistance to motion of the structure shown in Figure 8(b), including all resist-

ing elements. Similarly, this analysis must be repeated for all floors and all directions of interest.

When performing the analysis indicated above, the engineer must take into account any structural details that could affect his analysis, such as connections that will not resist movement, the cracking of concrete sections, instability, and limits to the amount of deflection allowed before linear behavior ends. He must also consider the P-Δ effect; this is the effect of axial loads (P) which have been displaced some distance (Δ) and then contribute to overturning moments, which in turn contribute to more deflection, etc. This is shown in Figure 8(c). In general, the P-Δ effects may be ignored if the resulting moments are less than 10 percent of the other design moments.

Since it is long and repetitious, the analysis is usually performed by a computer. However, the engineer must be able to input all of the values necessary to perform his analysis, such as the cross-sectional area, the modulus of elasticity, the moment of inertia in both directions, and any details in the structure such as hinges.

Assignment of Damping Values

A present, less is known about the damping values to use in real structures than about other parts of these analysis techniques. Considerable research is underway to develop better understanding of this effect and to provide a more rational way of determining correct values for various structures.

Pending the results of this research, it is appropriate to perform several analyses of the structure with various values for the damping factor. Currently 0, 0.5, 2, 5, and 10 percent of critical damping are appropriate. While this gives five separate results, it at least brackets the response the engineer should expect from the real structure.

Calculation of Mode Shapes and Frequencies

After the structure has been modeled, the analyst should determine the mode shapes and frequencies of the system. An exact method would involve the solution of a series of differential equations, a procedure that is described in several references such as *Fundamentals of Vibration Analysis*[3] by Myklestad and *Vibration Problems in Engineering*[4] by Timoshenko. This becomes quite tedious if done by hand for systems with several degrees of freedom. The Rayleigh method,[5] an approximate method which will give satisfactory accuracy for the first few modes, is also available. If more than one or two modes are required, however, the accuracy necessary in these hand calculations prohibits their use as a design method. Consequently, this calculation is usually accomplished with the computer and standard programs.

Number of Modes to be Considered

At this point it should be recalled that a simple-looking structure will have many frequencies and mode shapes, since each mass and resisting element has its own impact on the total system. Even if the grossest simplifying assumptions are made, there will be at least three motions for each mass—the x direction, the y direction, and the z direction (Figure 9). There will usually be torsion about these axes, but for most structures, these torsional effects are not emphasized. Thus, even for a three-story building there will be at least nine degrees of freedom, and thus nine mode shapes and frequencies.

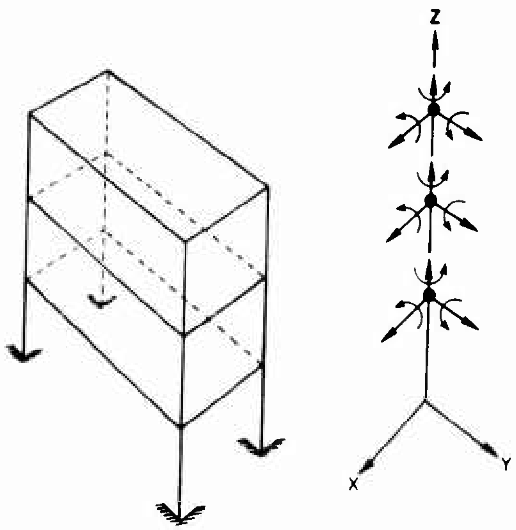

Figure 9. Possible degrees of freedom for a three-story structure.

[3] N. O. Myklestad, *Fundamentals of Vibration Analysis* (McGraw-Hill, 1956).

[4] Stephen Timoshenko, *Vibration Problems in Engineering* (Van Nostrand, 1955).

[5] Myklestad.

Theoretically, for a symmetrical structure these motions could be uncoupled and analyzed separately. Practically, however, because of nonuniform materials, construction, loadings, and ground motions these effects are interrelated and cannot be considered separately without introducing some error. Depending upon the type and design of the structure, this error may be relatively small and the structure may be uncoupled and analyzed separately in the three principal directions.

"L"-shaped structures or those with the center of mass significantly displaced from the center of resistance (5 percent of the major dimension in the plan view of the structure) should be analyzed with motions in all directions considered at the same time; the effects cannot be uncoupled.

With a normal, fairly symmetrical structure, the motions can be decoupled and effects considered in only one direction at a time. Even so, as many mode shapes and frequencies as there are floors in the structure must be considered. Fortunately, not all of the frequencies and mode shapes need to be evaluated to have an adequate evaluation of a structure because:

(1) The participation factors for the high frequencies are usually much smaller than those for the first three or four modes.

(2) With higher frequencies the masses must move faster. Since the damping force is directly proportional to the velocity, these higher mode responses are damped out faster.

(3) As shown in Figure 6, as the frequency goes up (or the period goes down) the responses of the higher modes to the typical earthquake in terms of velocity and displacement are insignificant, and the acceleration responses approach a constant value.

Thus analysis can usually be confined to the first three or four modes, or the ones that have frequencies $\leqslant 20$ cps.

Calculation of the Response

The analyst can now predict the response of a structure, assuming the structure remains linear or nearly linear in its response, and assuming he can in general predict the kind of motion expected. The methods are the response spectrum method and the time history methods described under "Description of Modal Analysis Method."

• Response Spectrum Method. In this method, the analyst must know the response spectrum for systems with periods similar to the periods of the structure being investigated. The expected responses of these modes in the real structure can then be computed using the appropriate participation factor, and combined by adding their absolute values or by taking the square root of the sum of their squares. As mentioned before, adding the absolute values gives an upper bound for the expected response, while using the square root of the sum of the squares yields a more probable value for the response. In cases where some of the modal frequencies are close together, the absolute sum may give a better result. These calculations can be done by computer or by hand.

• Time History Method. This method requires that the time history responses of single-degree-of-freedom systems having the same periods as the real structure be calculated for the entire duration of the input motion. The responses of these systems, multiplied by the participation factors for the mode shapes and the mode shape of the real structure itself, are then added together for each time interval to yield the time history response of the real structure. Since a very large number of calculations are required in this method, it can only realistically be done with a computer.

Accuracy and Applicability of Method to Various Structural Concepts

It is generally assumed that the response spectrum method of analysis described above is applicable to most structures, as long as they remain linearly elastic. However, in using this method with certain structural systems and configurations, the results may appear to be in error until they are reconciled with the behavior of the structure and the modeling assumptions. For example, if the structural system employs both shear walls and a moment resisting space frame, interactions between the two systems will occur; that is, in the lower stories, the shear wall will support the space frame and in the upper stories the space frame will support the shear wall. As a

result of this interaction, opposing forces will be present in a given story, leading to much larger values for shear in this transition zone than occur immediately above or below it. A similar observation has been noted where structural parts of widely divergent stiffnesses have been joined together, such as building setbacks. Thus, while the response spectrum method yields accurate results, these results sometimes lead to questions about the appropriateness of the results; time history analysis is then required for clarification.

Another limitation of the response spectrum method is that only maximum or expected maximum values of a function result from the analysis. Some equipment or mechanical or structural components attached to the structure may need to be evaluated independently for their response. This can only be accomplished if the time history response of the structure itself is known. In effect, the structural motion becomes the earthquake and the component becomes the structure in an analogy to the structural response calculations.

A further complication in the response spectrum method is that while the sum of the absolute values of responses predicts an upper bound for a function, the square root of the sum of the squares often predicts a value of the function which is lower than the value calculated from a time history evaluation (Table 2).

Table 2

Shear (in kips) for 10-Story Hospital Subjected to Typical Earthquake*

Story	Time History Analysis	Spectral Analysis $(F_n X_n \mu_n)^2$	1968 SEAOC Code Design Value
10	7,517	6,511	725
9	11,675	10,085	1,194
8	17,246	14,080	1,546
7	20,547	16,470	1,780
6	23,090	18,380	1,899
5	24,894	20,600	2,919
4	25,890	22,760	3,779
3	30,896	29,020	6,429
2	37,559	32,780	7,814
1	39,000	33,290	8,554

*Agbabian Associates, *Existing Capacity and Strengthening Concepts for Letterman and Hays Hospital (Task 8)*, draft report (Construction Engineering Research Laboratory, April 1974).

This phenomenon has been noted most often when there are few degrees of freedom. As the number of degrees of freedom increases, the time history response approaches the values from the spectral analysis. While neither method can give exactly accurate results, the discrepancy shown in the table indicates that the spectral analysis (square root of the sum of the squares) method should be used with extra care when there are only a few degrees of freedom involved in the response.

While the above discussion indicates that time history response may be necessary in some cases, it should be recognized that this evaluation is expensive due to the considerable cost of computer time necessary for the calculations and the time needed to evaluate and assimilate the results.

When performing any of these calculations, the analyst must not become enamored by the numbers that result. While the computer can produce numbers to six or seven figures, only about one or perhaps two digits at most are significant. A great many assumptions are involved in the modeling of the structure and the use of past earthquakes to predict future ground motions. Even the time history modal response calculations are no more accurate than the assumptions used in setting up the scenario and modeling the structure. While continued research is expected to improve this situation, considerable engineering judgment is currently required in using the results of any of these evaluations.

3 SUMMARY

From the discussion above, it can be seen that the modal analysis method can be applied to most structures provided realistic, simplifying assumptions are made and engineering judgment is used in evaluating the results. In general, with the uncertainties involved in understanding the energy dissipation in the structure and in predicting future motions, it appears unreasonable to try to be very accurate with most analyses. Rather, the response spectrum method provides sufficient accuracy for the general case using approximately the first four modes for evaluation. If, however, the structure is more complicated in its concept—it has reentrant corners, large setbacks, or has a combination of structural

systems with widely different stiffnesses—a more detailed time history analysis may be justified. Additionally, a time history response should be accomplished if a detailed analysis of the response of supported equipment or subsystems is required.

In either case, access to a computer with relatively large capacity is essential, although much more computer time is required for the time history response than for the response spectrum approach.

4 DESIGN RECOMMENDATIONS

For most evaluations, it is recommended that the response spectrum method be used to investigate structures subjected to earthquake motions. For structures having reentrants, large setbacks, combinations of structural support systems, or complicated equipment systems, the time history method of analysis should be considered.

APPENDIX A

REVIEW OF MODAL ANALYSIS THEORY

This Appendix reviews the theory associated with modal analysis methods and amplifies the discussion in the main body of the report.

Single-Degree-of-Freedom System

In the analysis of structural dynamic systems—structural systems subjected to vibratory motion—the basic building block is the single-degree-of-freedom system (Figure A1). This system includes a base or support system, a mass which moves or vibrates, a stiffness or restoring system which returns the mass to its undeflected position, and a damping or energy-absorbing system.

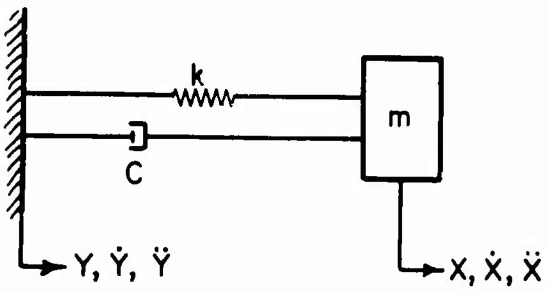

Figure A1. Typical single-degree-of-freedom system.

Figure A1 also shows a coordinate system in which X, $\dot{X}$, and $\ddot{X}$ represent the absolute deflection, velocity, and acceleration of the mass respectively and Y, $\dot{Y}$, and $\ddot{Y}$ similar values for the base.

Assume for the moment that the base is stationary (i.e., $Y = \dot{Y} = \ddot{Y} = 0$) and that some force has displaced the mass from its at-rest position. Also assume that the damping is directly proportional to the relative velocity of the mass with respect to its base. Then if X and $\dot{X}$ are positive, the forces shown in Figure A2 act on the mass.

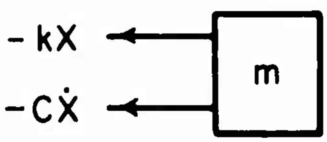

Figure A2. Forces acting on the mass for single-degree-of-freedom system.

Using Newton's Second Law (F = ma) and summing forces gives

$$-kX - c\dot{X} = m\ddot{X}, \text{ or} \qquad \text{[Eq A1]}$$

$$m\ddot{X} + c\dot{X} + kX = 0 \qquad \text{[Eq A2]}$$

Dividing by m and letting $\omega^2 = k/m$ and $\zeta = c/2\omega m$ gives

$$\ddot{X} + 2\zeta\omega\dot{X} + \omega^2 X = 0 \qquad \text{[Eq A3]}$$

A general solution for this, appropriate to structural dynamic problems, takes the form

$$X = e^{-\zeta\omega t}\left[A \cos\left(\sqrt{1-\zeta^2}\right)\omega t + B \sin\left(\sqrt{1-\zeta^2}\right)\omega t\right]$$

$$\text{[Eq A4]}$$

If there is no damping in the system ($\zeta = 0$), then Eq A4 reduces to

$$X = A \cos \omega t + B \sin \omega t \qquad \text{[Eq A5]}$$

or

$$X = C \sin(\omega t + a) \qquad \text{[Eq A6]}$$

This is the equation of free vibration in which C is the amplitude of motion, ω is the circular natural frequency (radians/second), and a is a phase angle dependent upon the initial conditions of motion. The vibration represented by Eq A6 is shown in Figure A3. The period of the system is the time it takes to make one complete cycle, ($T = 2\pi/\omega$). The frequency is the number of complete cycles per second ($f = \omega/2\pi = 1/T$).

If ζ is not equal to zero in Eq A4, then damping exists in the system, and the motion can be represented by Figure A4. In this case, the motion is gradually damped out. The damped circular frequency (ω_d) is now $\omega\sqrt{1-\zeta^2}$, and the damped period T_d is $2\pi/\omega\sqrt{1-\zeta^2}$. Experience with real structures indicates that they have values of ζ equal to or less than 0.2 (20 percent damping). Thus there is minimum effect on the frequency ($\omega_d \geqq 97.98$ percent of ω) and the period ($T_d \leqq 1.02T$). For all practical purposes, the ω and T for the undamped systems can be used to represent the frequency and period of the damped systems.

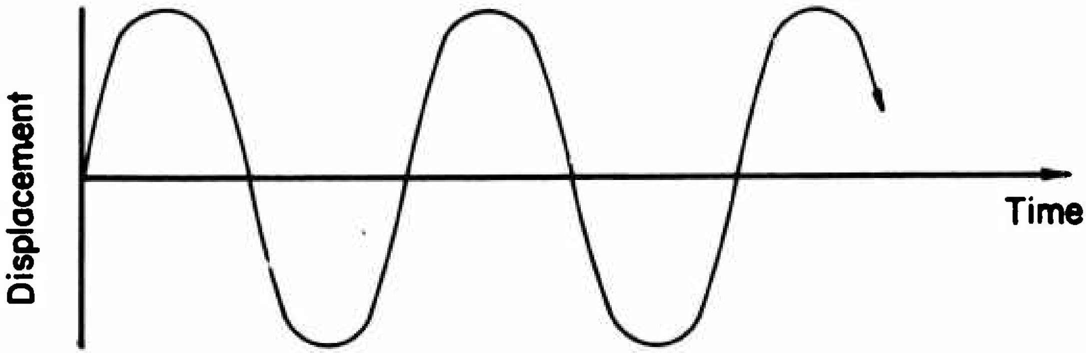

Figure A3. Plot of displacement versus time for undamped single-degree-of-freedom system.

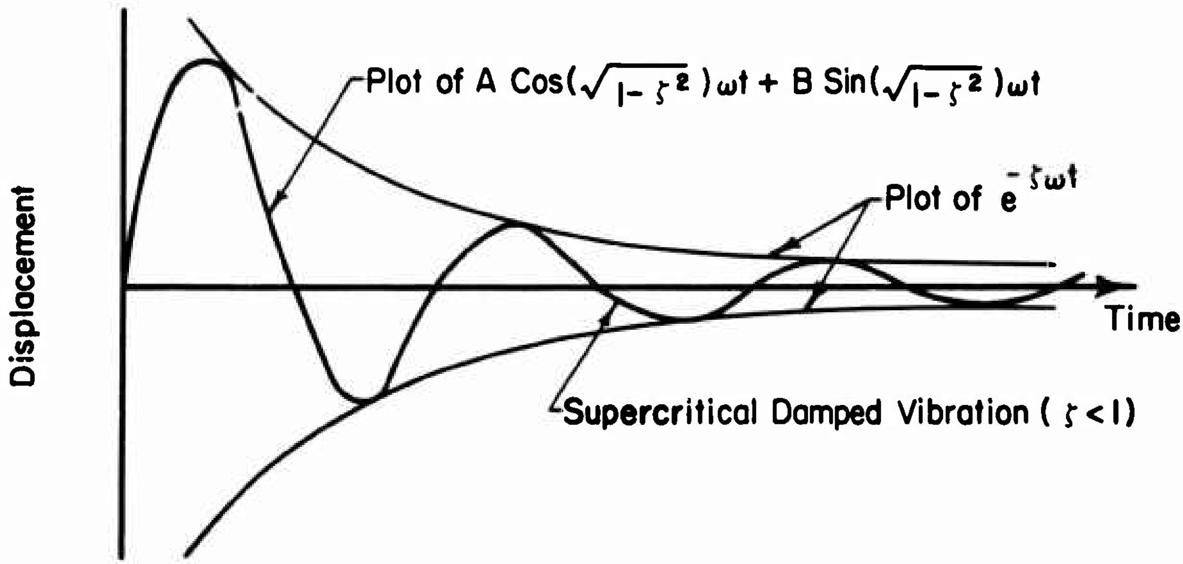

Figure A4. Plot of displacement versus time for subcritically damped single-degree-of-freedom system.

If the values of C and ω are known in Eq A6, then the maximum values of X, $\dot{X}$, and $\ddot{X}$ are also known:

$$X_{max} = C \qquad \text{[Eq A7]}$$

$$\dot{X}_{max} = \omega C \qquad \text{[Eq A8]}$$

$$\ddot{X}_{max} = \omega^2 C \qquad \text{[Eq A9]}$$

An understanding of these relationships is essential when dealing with modal analysis methods.

Returning to Eq A3, it should be remembered that $\omega^2 = k/m$ or the circular natural frequency $\omega = \sqrt{k/m}$. If k is increased or m decreased, ω and thus the maximum values of $\dot{X}$ and $\ddot{X}$ in Eq A7 are increased. Conversely, if k is decreased or m increased, ω and thus the maximum values of $\dot{X}$ and $\ddot{X}$ are decreased.

Discussion of Damping of Single-Degree-of-Freedom Systems

All vibrating systems lose energy in some way; this is called damping of the system. Energy can be lost from a structural system in several different ways. For example, if two surfaces are rubbed together, coulomb, or dry friction, damping occurs. If the damping force is proportional to the relative velocity of the mass, viscous damping occurs. In some systems, damping is proportional to the square of the relative velocity; other concepts for damping are also used.

In real structures, the total damping is a combination of different types of damping, but damping always resists the relative motion of the mass. From the standpoint of theoretical analysis of these systems, the easiest type of damping to deal with is the one in which the resisting force is a linear function of the velocity. Experience has shown that this assumption results in reasonably accurate results, and it is the kind of damping that is used in this report.

The damping factor ζ used in Eq A3 and subsequent work equals $c/2\omega m$, and is a dimensionless number giving a multiple (or fraction) of critical damping. Critical damping is defined as the value of damping for the system which represents the boundary between cyclic and aperiodic behavior of the system. If $\zeta < 1$, then cyclic response occurs (Figure A4). If $\zeta > 1$, then aperiodic motion results (Figure A5).

As mentioned above, realistic values of ζ for structural systems are equal to or less than 0.2. Values less than 0.1 are usually applicable.

Multi-Degree-of-Freedom Systems

Almost all systems of interest are more complicated than the single-degree-of-freedom system described above. Consequently, this section describes a method of analyzing more complex systems. Figure A6 shows a rather simple multi-degree-of-freedom system which is representative of most structures. Some structures, such as nuclear reactors, are much more complicated in that mass two might be connected with the base as well as with masses one and three. Other complications are also quite possible, but for the general building, Figure A6 is acceptably representative.

In Figure A6, each mass has only one component of motion; therefore, this system has three degrees of freedom, three frequencies, and three mode shapes. In a real structure, each mass usually has the ability to move in three orthogonal directions and twist about these three orthogonal axes. If this were true

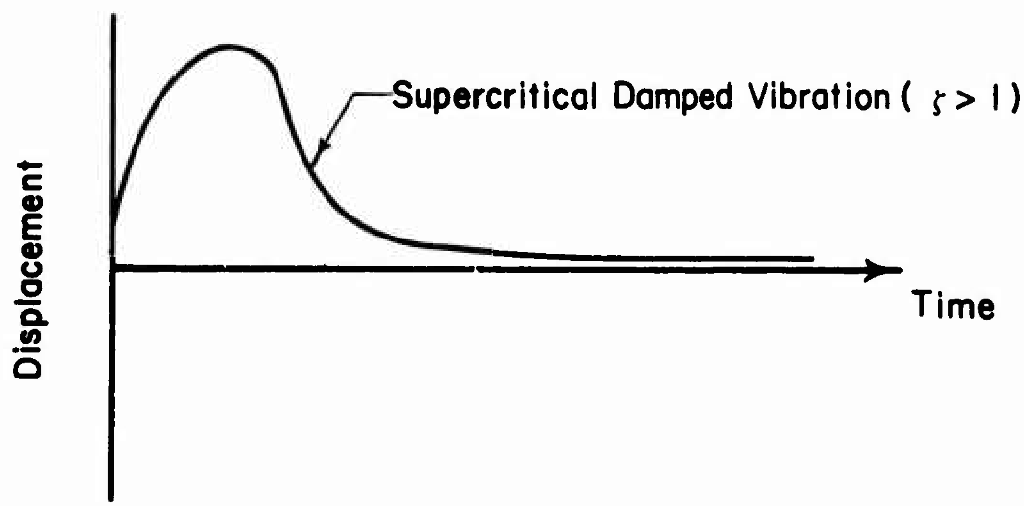

Figure A5. Plot of displacement versus time for supercritically damped single-degree-of-freedom system.

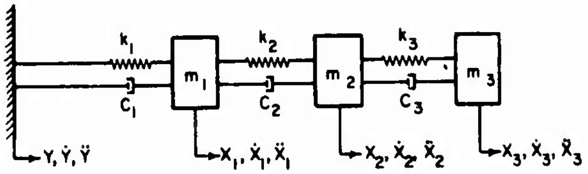

Figure A6. Typical multi-degree-of-freedom system.

for Figure A6, the system would have 18 degrees of freedom with 18 frequencies of vibration and 18 separate mode shapes.

Generally, this complication does not present a significant problem, since the responses of a structure in the various mode shapes can be decoupled and treated separately.

To find the natural frequencies and mode shapes, the same type of analysis is performed as was done for single-degree-of-freedom systems. As a starting point, assume that the base is stationary and that $X_3 > X_2 > X_1$ with forces shown in Figure A7. Using Newton's Second Law and summing forces gives

$$-k_1X_1 + k_2(X_2 - X_1) = m_1\ddot{X}_1 \qquad \text{[Eq A10]}$$

$$-k_2(X_2 - X_1) + k_3(X_3 - X_2) = m_2\ddot{X}_2 \qquad \text{[Eq A11]}$$

$$-k_3(X_3 - X_2) = m_3\ddot{X}_3 \qquad \text{[Eq A12]}$$

Rewriting these gives

$$m_1\ddot{X}_1 + (k_1 + k_2)X_1 - k_2X_2 = 0 \qquad \text{[Eq A13]}$$

$$m_2\ddot{X}_2 - k_2X_1 + (k_2 + k_3)X_2 - k_3(X_3) = 0 \qquad \text{[Eq A14]}$$

$$m_3\ddot{X}_3 - k_3X_2 + k_3X_3 = 0 \qquad \text{[Eq A15]}$$

Since these equations are similar to Eq A2, solutions are expected of the form

$$X_1 = D_1 \sin(\omega t + a) \qquad \text{[Eq A16]}$$

$$X_2 = D_2 \sin(\omega t + a) \qquad \text{[Eq A17]}$$

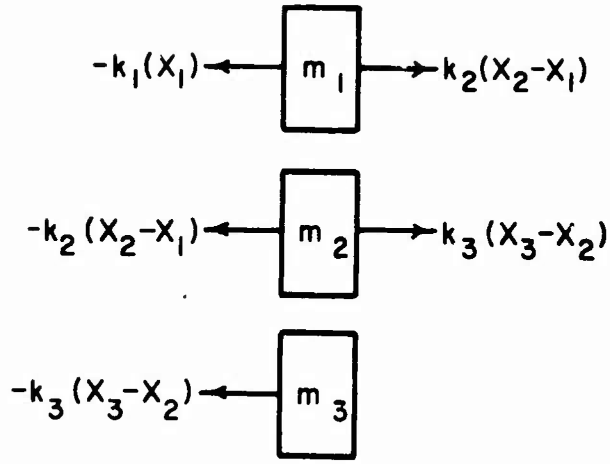

Figure A7. Forces acting on masses for three-degree-of-freedom system $C_1 = C_2 = C_3 = 0$ and $X_3 > X_2 > X_1$.

$$X_3 = D_3 \sin(\omega t + a) \qquad \text{[Eq A18]}$$

Substituting these values and their derivatives into Eq A13, A14, and A15 gives

$$\sin(\omega t + a)\{[-m_1\omega^2 + (k_1 + k_2)]D_1 - k_2D_2\} = 0 \qquad \text{[Eq A19]}$$

$$\sin(\omega t + a)\{-k_2D_1 + [-m_2\omega^2 + (k_2 + k_3)]D_2 - k_3D_3\} = 0 \qquad \text{[Eq A20]}$$

$$\sin(\omega t + a)\{-k_3D_2 + [-m_3\omega^2 + k_3]D_3\} = 0 \qquad \text{[Eq A21]}$$

For these equations to be true for all values of t and to have nontrivial solutions ($D_1 \neq 0$, and $D_3 \neq 0$), the following determinant must be equal to zero:

$$\begin{vmatrix} [(k_1 + k_2) - m_1\omega^2] & -k_2 & 0 \\ -k_2 & [(k_2 + k_3) - m_2\omega^2] & -k_3 \\ 0 & -k_3 & [k_3 - m_3\omega^2] \end{vmatrix} = 0 \qquad \text{[Eq A22]}$$

Performing this evaluation yields three values of ω^2 in terms of k/m, which when evaluated will give the three natural frequencies of vibration for the system.

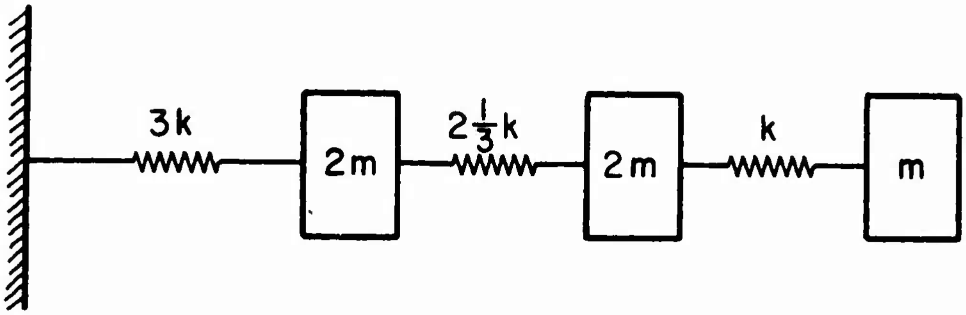

Figure A8. Example three-degree-of-freedom system.

As an example, take a three-story structure and assume that the mass of the second and third floor is twice the mass of the roof. Using a typical code, the relative stiffnesses would be as shown in Figure A8.

Substituting these values into Eq A22 gives

$$\begin{vmatrix} (5\frac{1}{3}k - 2m\omega^2) & -2\frac{1}{3}k & 0 \\ -2\frac{1}{3}k & (3\frac{1}{3}k - 2m\omega^2) & -k \\ 0 & -k & (k - m\omega^2) \end{vmatrix} = 0$$

Solving for ω^2 gives $\omega_1^2 = \frac{k}{3m}$

$$\omega_2^2 = \frac{3k}{2m}$$

$$\omega_3^2 = \frac{7k}{2m}$$

or

$$\omega_1 = \sqrt{\frac{k}{3m}} = 0.5774\sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{\frac{3k}{2m}} = 1.2247\sqrt{\frac{k}{m}}$$

$$\omega_3 = \sqrt{\frac{7k}{2m}} = 1.8708\sqrt{\frac{k}{m}}$$

To find the mode shape, values for ω, m, and k are substituted in Eq A19, A20, and A21. While exact values are not available, values for D_2/D_1 and D_3/D_1 are. Assuming that D_1 is equal to some unit displacement, a representative mode shape can be obtained. For the example, the following values result:

	First Mode	Second Mode	Third Mode
D_1	1	1	1
D_2	2	1	$-\frac{5}{7}$
D_3	3	-2	$\frac{2}{7}$

These mode shapes are shown in Figure A9.

Referring to Eq A16, A17, and A18, it can be seen that these mode shapes represent relative values only and that the displacements for any mode n at any time t are

$$X_1 = D_1 \sin(\omega_n t + a) = 1[D_1 \sin(\omega_n t + a)] \quad \text{[Eq A23]}$$

$$X_2 = D_2 \sin(\omega_n t + a) = \frac{D_2}{D_1}[D_1 \sin(\omega_n t + a)] \quad \text{[Eq A24]}$$

$$X_3 = D_3 \sin(\omega_n t + a) = \frac{D_3}{D_1}[D_1 \sin(\omega_n t + a)] \quad \text{[Eq A25]}$$

As a multi-degree-of-freedom system with independent or uncoupled modes is subjected to a base disturbance, each mode responds in its own way. Since D_1, D_2, and D_3 above are arbitrary values, the total response of a system can be represented by Eq 5 (from main body of report) which is reproduced here:

$$X_m(t) = \sum_{n=1}^{n} F_n (D_{mn}) \sin(\omega_n t)$$

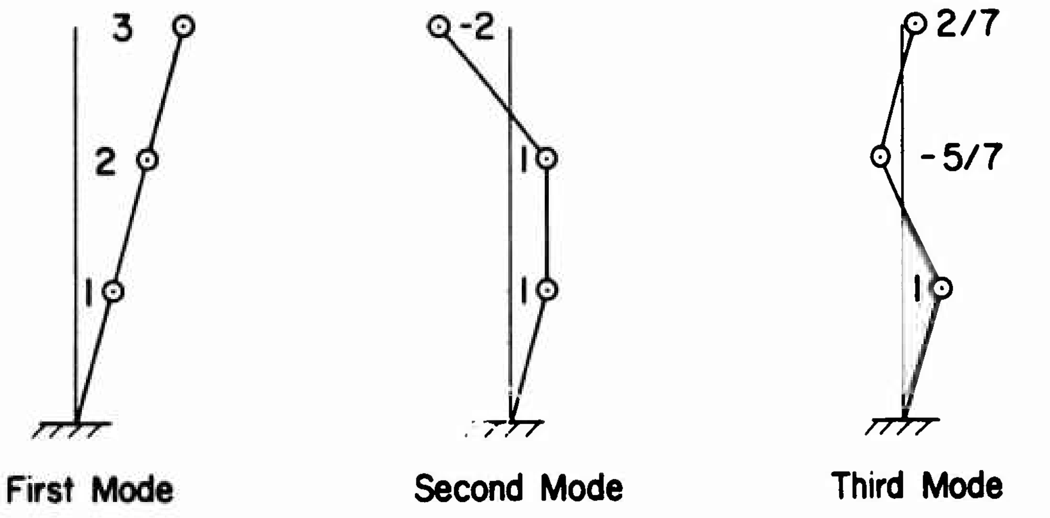

Figure A9. Principal mode shapes for example system.

where D_{mn} is the maximum displacement of mass m in mode n caused by the particular disturbance, and F_n is the participation of mode n in the total response. The phase angle *a* has been omitted since it does not affect the magnitude of the response.

It has been shown[6] that for earthquake motions, Eq A26 allows evaluation of F_n:

$$F_n = \frac{\sum_{i=1}^{j} m_i D_i}{\sum_{i=1}^{j} m_i (D_i)^2} \qquad \text{[Eq A26]}$$

where j represents the total number of degrees of freedom of the system. For the example, the values for the participation factor F_n are as follows:

$$F_1 = \frac{2m(1) + 2m(2) + m(3)}{2m(1)^2 + 2m(2)^2 + m(3)^2} = \frac{9m}{19m} = \frac{36}{76}$$

$$F_2 = \frac{2m(1) + 2m(1) + m(-2)}{2m(1)^2 + 2m(1)^2 + m(-2)^2} = \frac{2m}{8m} = \frac{19}{76}$$

$$F_3 = \frac{2m(1) + 2m(-{}^5/_7) + m({}^2/_7)}{2m(1)^2 + 2m(-{}^5/_7)^2 + m({}^2/_7)^2} = \frac{\frac{6m}{7}}{\frac{152m}{49}} = \frac{21}{76}$$

If other arbitrary values for D_1 are judiciously chosen, values for all F_n equal to 1 could result. This would simplify analysis methods.

While the discussion above is based on displacement alone, similar relationships exist for any linear function, such as shears, moments, rotations, accelerations, etc.

Damping of Multi-Degree-of-Freedom Systems

As mentioned earlier, damping is a complicated phenomenon. To understand it and its effects on multi-degree-of-freedom systems, defining what a mode is when damping is involved is necessary.

In the previous section, mode shapes were described; it was found that with multi-degree-of-freedom systems, the masses vibrate in such a way that the ratios of their displacements are always the same. Assume that

$$X_1 = D_1 \sin(\omega t \quad a)$$

$$X_2 = D_2 \sin(\omega t \quad a)$$

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[6] G. W. Housner, "Earthquake Resistant Design Based on Dynamic Properties of Earthquakes," *Proceedings*, Vol 53 (American Concrete Institute, 1956-57), pp 85-98.

$$X_3 = D_3 \sin(\omega t - a)$$

then

$$\frac{X_3}{X_1} = \frac{D_3 \sin(\omega t - a)}{D_1 \sin(\omega t - a)} = \frac{D_3}{D_1}$$

which is a constant for any particular frequency. Similarly,

$$\frac{X_2}{X_1} = \frac{D_2}{D_1} = \text{a constant.}$$

Extrapolating this to damped systems, for mode shapes to exist with damping, the ratios of the displacements of the masses should always be the same. If ζ = portion of critical damping for one frequency, ω = that frequency, and ω_d = the damped frequency ($\omega_d = \omega\sqrt{1 - \zeta^2}$), then the following relationships would exist

$$X_1 = D_1 e^{-\zeta\omega t} \sin(\omega_d t - a)$$

$$X_2 = D_2 e^{-\zeta\omega t} \sin(\omega_d t - a)$$

$$X_3 = D_3 e^{-\zeta\omega t} \sin(\omega_d t - a)$$

Then

$$\frac{X_3}{X_1} = \frac{D_3 e^{-\zeta\omega t} \sin(\omega_d t - a)}{D_1 e^{-\zeta\omega t} \sin(\omega_d t - a)} = \frac{D_3}{D_1} = \text{a constant}$$

and

$$\frac{X_2}{X_1} = \frac{D_2}{D_1} = \text{a constant for any particular system.}$$

Substituting these values into the equations of motion for the masses and solving, it can be shown that for modal response to be appropriate with interfloor damping,

$$\frac{c_1}{k_1} = \frac{c_2}{k_2} = \frac{c_3}{k_3} = \text{a constant for a particular system}$$

with a particular portion of critical damping. It can also be shown that

$$\frac{\zeta_1}{\omega_1} = \frac{\zeta_2}{\omega_2} = \frac{\zeta_3}{\omega_3} = \text{a constant.}$$

Once the portion of critical damping for one mode of a system is established, it is fixed for all other modes for that system, since the ω_n values are dependent only upon the masses and stiffnesses for that system.

It can also be shown from this work that the value of the damping coefficient (c) is established from the relationship

$$c_i = \frac{2k_i\zeta_n}{\omega_n}$$

where i refers to the relative location in the system and n refers to the frequency and mode numbers. Having established or selected the damping fraction for one frequency, and knowing all the k's and n's for the system, all of the damping coefficients are determined and the damping fraction for each of the other modes is fixed.

The above discussion is based upon theory. In practice, however, most computer programs allow the separate establishment of damping factors for each mode. This is considered appropriate for real structures for the following reasons:

(1) Structures generally have six degrees of freedom for each mass; deflection in the x, y, and z directions; and rotation about each of these axes. The stiffnesses and damping coefficients in these various directions need not be related to each other. Therefore, no relationship between the damping appropriate to one frequency of a structure and that of another is necessary.

(2) Viscous damping is only an idealization of the actual damping that exists. Some forms of actual damping depend upon deflection only and are not functions of velocity. Therefore, judgment should be used in applying the theory discussed above; increasing the portion of critical damping for the higher frequencies may not be necessary.

(3) Generally, increasing the damping results in less displacement and less response. Therefore, using smaller amounts of damping for the higher modes is conservative and does not result in large errors in analysis.

In summary, modal analysis can be used with damped multi-degree-of-freedom systems. Results will be sufficiently accurate even though percentages

of critical damping for each mode are arbitrarily established (using experience and engineering judgment) independent of the frequency relationships. Damping in these systems will usually not exceed 10 percent of critical damping.

APPENDIX B

SIGNIFICANCE AND USE OF RESPONSE SPECTRA

The response spectrum, a standard tool used in the analysis of vibrating systems, is usually a graph or plot of the expected responses of systems to a certain input motion. Figure 6 shows a typical earthquake response spectrum. In some cases, the frequency (f) is plotted along the horizontal axis, while in others the period of the system (T) is the horizontal coordinate, $f = 1/T$. Thus some spectra appear to be the mirror image of others, while in fact they are the same except they have a different coordinate system.

It can be theoretically shown that the response of an undamped system to a particular motion is a function of the motion itself and the natural frequency of the system. While this seems to neglect the stiffness and mass of the system, it does not, since the frequency is directly proportional to the square root of the stiffness and inversely proportional to the square root of the mass,

$$\omega = \sqrt{\frac{k}{m}}$$

Thus if the stiffness and the mass of a system are doubled, the frequency remains the same and the same response can be expected for the new system as for the old.

In general, response spectra are prepared by calculating the response of single-degree-of-freedom systems with various amounts of damping to measured earthquake motions. Mathematical integration methods are available which apply the measured motions to the base of a system, with integration over short time intervals, and calculation of the response of the mass. They proceed in a step-by-step process until the total earthquake record has been completed. The largest value of the function of interest is recorded and becomes the response of that system to that motion. Changing the parameters of the system to change the frequency, the process is repeated and another response recorded. This process is repeated until all frequencies of interest have been covered and the results plotted. This becomes the response spectrum for that motion. Since no two earthquakes are alike, this total process must be repeated for all earthquakes of interest.

Since the time of the maximum response is not recorded in this procedure, when the maximum response occurs is unknown.

Until relatively recently, there were few recorded earthquake motions because there were few accelerometers emplaced to measure them; the El Centro, CA, earthquake of 1940 was the most severe earthquake recorded and was used as the basis for much analytical work. Recently, however, many more earthquakes have been measured. Maximum recorded accelerations have gone from about 0.32 g for the El Centro earthquake to values greater than 0.5 g. It is expected that even larger values will be measured as more instruments are placed closer to epicenters of active earthquakes.

Earthquakes consist of a series of random ground motions. Usually the north-south, east-west, and vertical components of accelerations are measured. Currently, no accurate predictive method has been developed which allows description of the particular motion that a site can be expected to experience. Thus, it is better to use a consolidated response spectrum which incorporates the consolidated spectra for several earthquakes with the primary variable, as far as the earthquake motion is concerned, being the maximum acceleration. Such a prediction is shown in Figure B1. Also shown in that figure are the maximum responses expected when various amounts of critical damping are applied to the system.

Three generalizations can be made about the response spectrum shown in Figure B1. When the frequency is low (less than 0.2 cps), the displacement response is fairly constant. This corresponds to a system having a relatively small stiffness with respect to its mass. When the frequency is large (greater than 2 cps), the acceleration is relatively constant. This corresponds to a system having a large stiffness compared to its mass. In the midrange, the pseudo-velocity (or velocity of the mass relative to its base) remains fairly constant with a change in the frequency. This is consistent with the response expected from such systems.

The following is a demonstration of the use of the response spectrum method in conjunction with the example structure analyzed in Appendix A.

Assume that the value of relative stiffness (k) used

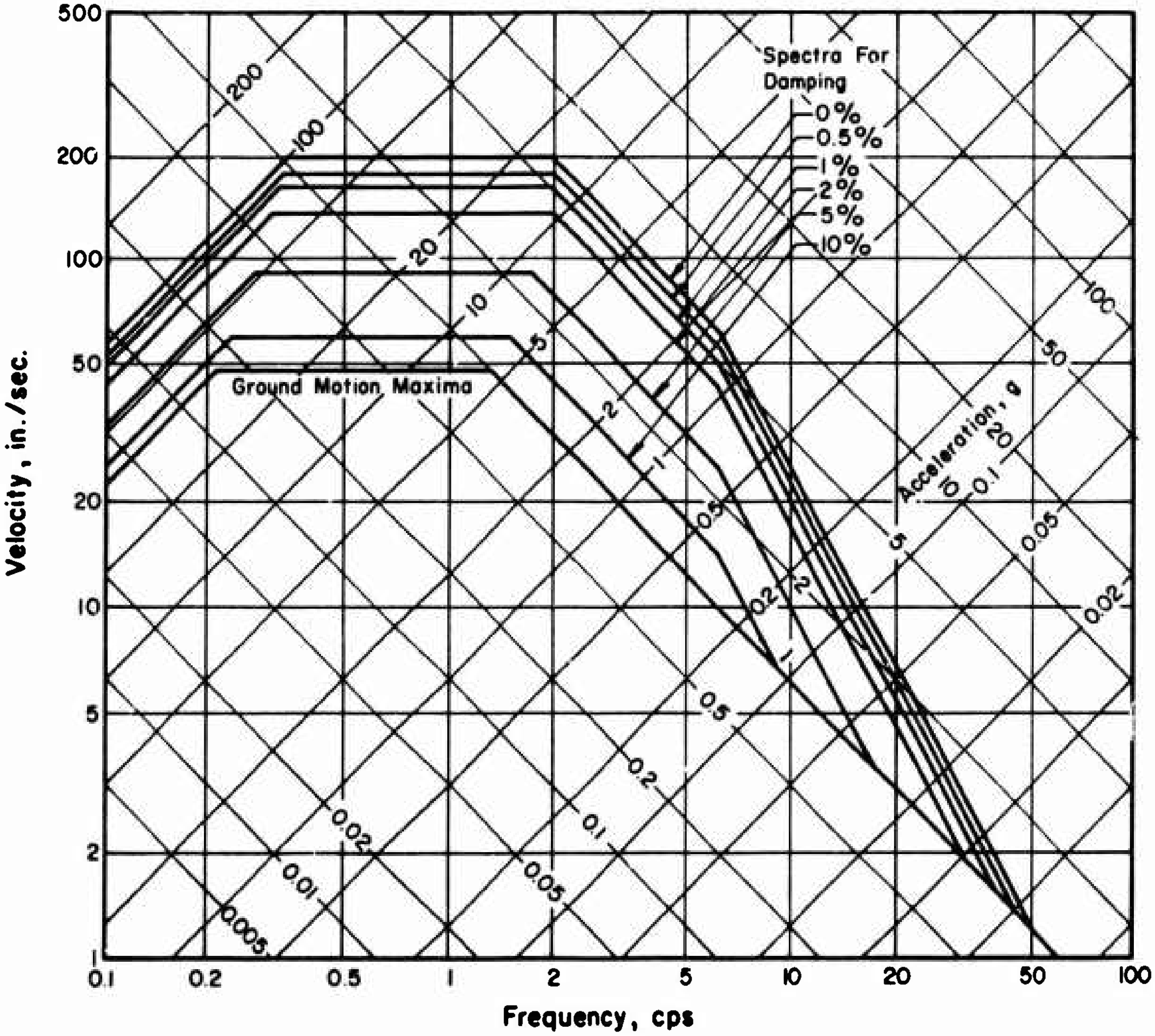

Figure B1. Basic design spectra normalized to 1.0 g. From N. M. Newmark and W. J. Hall, "Procedures and Criteria for Earthquake Resistant Design," *Building Practices for Disaster Mitigation* (Department of Commerce, February 1973).

in that example is 500 kips/in., and the value of the relative mass (m) is 2 kip-seconds²/in. This mass is equivalent to a weight of about 773 kips. Then the frequencies will be

$$\omega_1 = 0.5774\sqrt{\frac{k}{m}} = .5774\sqrt{\frac{500}{2}} = 9.1295\ \frac{\text{radians}}{\text{second}}$$

$$= 1.45 \text{ cps}$$

$$\omega_2 = 1.2247\sqrt{\frac{k}{m}} = 1.2247\sqrt{250} = 19.3642 \text{ rps}$$

$$= 3.08 \text{ cps}$$

$$\omega_3 = 1.8708\sqrt{\frac{k}{m}} = 1.8708\sqrt{250} = 29.5799 \text{ rps}$$

$$= 4.71 \text{ cps}$$

Assuming zero damping with these frequencies and using Figure B1, displacements of 22.0, 6.8, and 2.9 in. for modes 1, 2, and 3 respectively result.

Using these values in conjunction with the modal displacements and participation factors calculated in Appendix A gives the following values for the modal responses:

	First Mode	Second Mode	Third Mode
Displacement, mass one	10.4 in.	1.7 in.	0.8 in.
Displacement, mass two	20.8 in.	1.7 in.	– 0.6 in.
Displacement, mass three	31.3 in.	– 3.4 in.	0.2 in.

These displacements are relative to the base of the structure.

To predict the maximum displacement of this structure, either the first mode, the maximum absolute value, or the maximum expected value (the square root of the sum of the squares) can be used. The values obtained using the three figures are:

	First Mode	Maximum Absolute Value	Expected Value
Mass one	10.4 in.	12.9 in.	10.6 in.
Mass two	20.8 in.	23.1 in.	20.9 in.
Mass three	31.3 in.	34.9 in.	31.5 in.

If the relative deflections between the masses, the shears, or some other linear function of the structure are desired, it would be necessary to start with the correct functional value (relative deflection) but use the same participation factor and same value from Figure B1, combining them in the same way. For example, the acceleration responses would be 4.6 g, 6.5 g, and 6.5 g for modes 1, 2, and 3 respectively.

Using these values with the modal displacements and participation factors from Appendix A gives the following values for the modal accelerations:

	First Mode	Second Mode	Third Mode
Acceleration, mass one	2.27 g	1.63 g	1.80 g
Acceleration, mass two	4.55 g	1.63 g	– 1.28 g
Acceleration, mass three	6.82 g	– 3.25 g	0.51 g

Combining these in the same way as above gives:

	First Mode	Maximum Absolute Value	Expected Value
Mass one	2.27 g	5.70 g	3.32 g
Mass two	4.55 g	7.46 g	5.00 g
Mass three	6.82 g	10.58 g	7.57 g

The responses shown above appear to be quite large, partially because the design spectrum is normalized to a maximum ground acceleration of 1.0 g. If the maximum ground acceleration expected at a particular site is 0.3 g, then each of the values shown above would be multiplied by the factor 0.3, thus reducing the responses to realistic values. Additionally, small amounts of damping would significantly reduce the results, as can be seen from Figure B1.

The dominance of the first mode in the responses shown above should be noted.

In summary, the response spectrum method provides a fast, reasonably accurate tool to help in the analysis of structures subjected to earthquake motions.

APPENDIX C

EXTENSION OF THE MODAL ANALYSIS METHOD TO THE INELASTIC CASE

Most well designed and constructed buildings are able to withstand minor earthquakes with little or no damage. However, a structure built in seismic areas will generally be subjected to large or severe earthquakes sometime in its life span. While structures can be designed to resist these earthquakes, it is not economically feasible or realistic to design all buildings to withstand elastically the greatest possible earthquake. Thus, in general, the analyst must consider how his structure will respond inelastically to earthquakes he might reasonably expect it to experience.

Discussion of Inelastic Action in Vibrating Systems

While most structural materials behave elastically for small displacements, they sooner or later undergo inelastic action. The inelastic force-displacement relationship that is usually used in earthquake analysis is shown in Figure C1; it is referred to as elastoplastic action. If the force or deflection is removed prior to the occurrence of yielding, the material will return along its loading line to the origin. If the force continues long enough, or if the displacement exceeds the yield value, some permanent deformation will occur, and the structure will be permanently deformed unless it is subsequently deformed the same amount in the opposite direction.

In a typical single-degree-of-freedom system (Figure A1) that is responding elastically, the response continues to act along the elastic action line shown in Figure C1. This is the basis for the analysis contained in Appendix A. The force on the mass, or the measure of the acceleration, is directly proportional to the deflection. The period of the structure and the frequency are inversely related, and the energy of the system ($\frac{1}{2}$ mV^2) is related to the triangular area under the elastic action line.

With an inelastic system, these fundamental relationships do not hold true for the entire vibration. For instance, the force on the mass can never exceed the force which occurs when yielding occurs. Thus when yielding occurs, the acceleration is reduced from what it would be for the same system with the same deflection if the system remained elastic. Additionally, while the energy is still represented by the area under the force deflection line, only part of this work (represented by A in Figure C1), contributes to subsequent vibration. The rest of the work has been lost as far as vibrations are concerned. Conceptually, this is somewhat similar to the energy lost by damping.

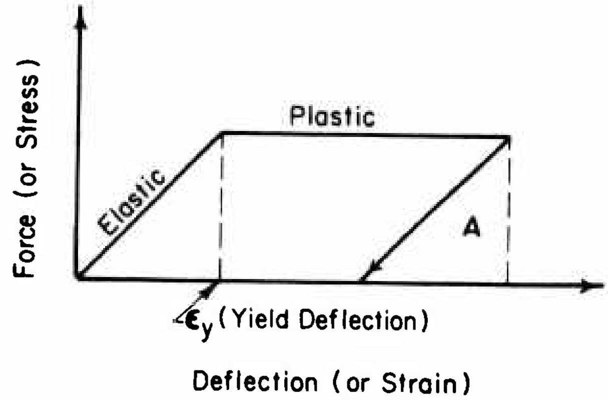

Figure C1. Force-displacement relationship for elastoplastic system.

As far as displacements are concerned, larger displacements might be expected since it takes a greater deflection under the inelastic condition to store the same amount of work as it does under the elastic condition. This does not appear to be the case in actual calculations for earthquakes, however. The energy lost in the inelastic action reduces the total energy available to the system for subsequent vibrations and thus reduces the deflections which are subsequently observed.

Comparison of Elastic and Inelastic Response Spectra

In Figure B1 the elastic response spectrum for a typical strong motion earthquake is reproduced. Also shown are the bounds of the ground displacement, velocity, and acceleration. As noted in Appendix B, some generalizations can be made about this spectrum. When the frequency of the structure is small—the mass is large with respect to the stiffness—the maximum deflection of the system is equal to the maximum ground deflection. This occurs because the mass remains relatively stationary as the ground moves; thus, the relative deflection approaches the value of the deflection of the ground.

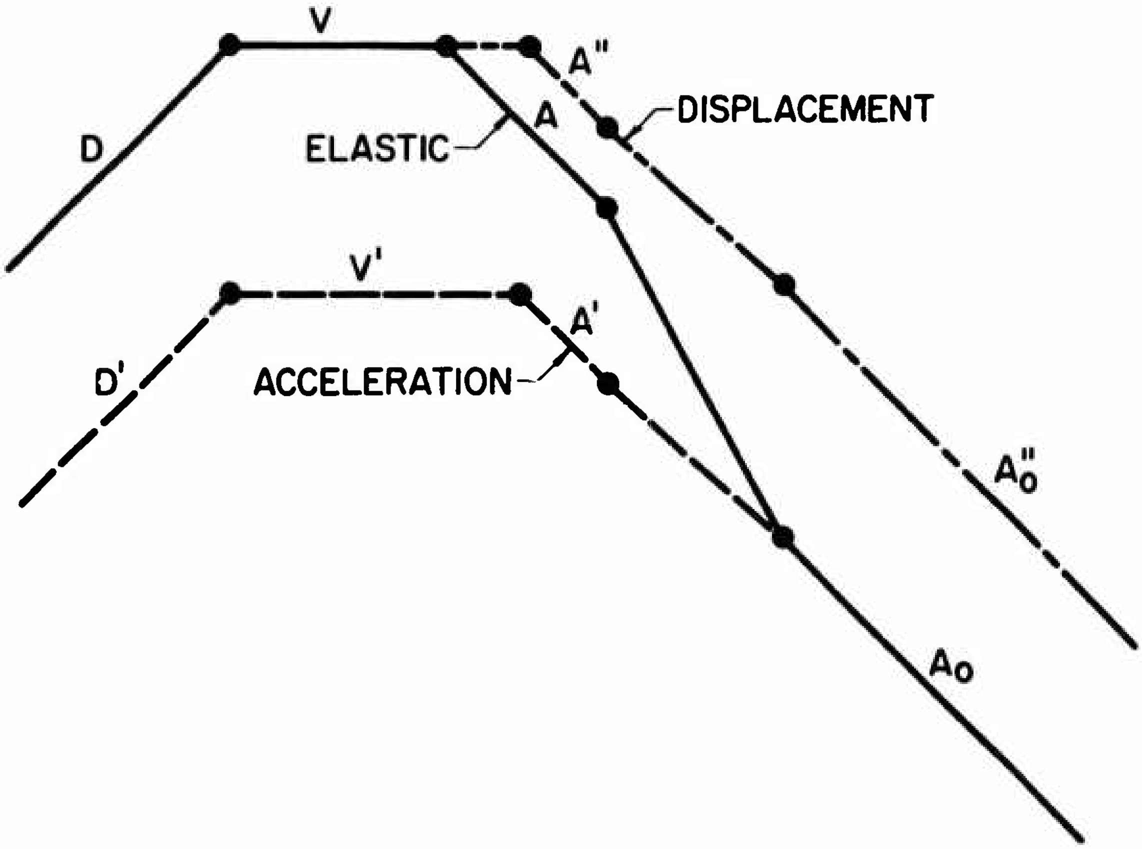

Figure C2. Design spectra. From N. M. Newmark and W. J. Hall, "Procedures and Criteria for Earthquake Resistant Design," *Building Practices for Disaster Mitigation* (Department of Commerce, February 1973).

At the other extreme, when the frequency is large—the mass is very small relative to the stiffness—the acceleration of the mass equals the acceleration of the ground. This occurs because the spring is so stiff that there is little relative deflection between the mass and its base, and the accelerations applied at the base are directly felt by the mass.

Between these two extremes is the region of amplification where for any one frequency, the displacement, velocity, and acceleration of the ground are amplified by the dynamic system.

Figure C2 shows a typical design spectrum currently recommended for use when inelastic action is anticipated. In this figure, the line D-V-A-A_0 represents the elastic spectrum shown in Figure B1.

For the reasons discussed above, the displacement, velocity, and acceleration spectral values are no longer related by $\dot{X}_{max} = \omega X_{max}$ and $\ddot{X}_{max} = \omega^2 X_{max}$, but certain of the relationships still apply. In the low frequency region the maximum displacement is still equal to the maximum ground deflection, and in the high frequency region the maximum acceleration is equal to the maximum ground acceleration. In between, however, there is a transition region that needs explanation. Before discussing this region, the relationship of displacement and acceleration at the extremes will be discussed.

In the low frequency region where the displacement is preserved, the acceleration of a structure is reduced. Since the force for an elastoplastic structure does not increase when yielding occurs, the

acceleration reaches a maximum value then also (Figure C1). Thus the acceleration is reduced by the factor $1/\mu$ where μ is the ratio of X maximum to X elastic. At the other end of the spectrum, it can be shown that the accelerations are preserved; that is, the maximum accelerations of the system equal the maximum ground accelerations, while the deflections are greater than the elastic deflections. Between these extremes, the energy in the system must be preserved; the lines in Figure C1 differ by a value $\sqrt{2\mu - 1}$ which is derived from conservation of energy methods.

In summary, the line D^1-V^1-A^1-A_0 represents the plot of maximum accelerations with inelastic action. The line D-V-A''-A_0'' represents the plot of maximum displacements with inelastic action, and the line D-V-A-A_0 represents the interrelated values of displacement and acceleration when elastic action applies.

Inelastic Modal Analysis

The application of modal analysis to inelastic structures is similar to that for elastic structures, except that for elastoplastic spectra the results can be used only as an approximation of the expected response.

One of the first values the designer must establish is the value of the ductility (μ) that will be allowed. For structures that must continue functioning after an earthquake, a value of 1.1 to 1.2 is appropriate. For values much larger than this, significant structural damage may result and the structure will not be functionally effective. While values up to 5 may be allowed before collapse occurs, the damage when $\mu >$ 1.2 to 1.3 will generally be too great to allow functional use after the ground motion.

Having established μ, the designer then prepares the response spectrum by using the μ factors and the elastic response spectrum. The elastic periods of the modes of interest are then calculated and the maximum value of the expected response from the spectrum determined. These values should then be used as they would be for an elastic evaluation. In this case, however, the designer should check whether the structure did in fact achieve the established μ factors. If it did not, the structure or μ factors should be modified and calculations redone.

In this way, an inelastic analysis of a structure can be performed.

Example

The following example illustrates the use of the design spectra in the inelastic case. Assume the same example structure used in Appendices A and B and the ground motion used in Appendix B. Also assume that a ductility factor of 1.5 is allowed, resulting in the design spectra shown in Figure C3.

The following results are achieved:

	Displacement	Acceleration
First mode	22.0 in.	3.2 g
Second mode	6.7 in.	4.5 g
Third mode	2.8 in.	4.5 g

Combining these as before gives the following displacements:

	First Mode	Maximum Absolute Value	Expected Value
Mass one	10.4 in.	12.9 in.	10.6 in.
Mass two	20.8 in.	23.1 in.	20.9 in.
Mass three	31.3 in.	34.9 in.	31.5 in.

and accelerations:

	First Mode	Maximum Absolute Value	Expected Value
Mass one	1.52 g	3.9 g	2.3 g
Mass two	3.03 g	5.1 g	3.4 g
Mass three	4.55 g	7.2 g	5.1 g

The displacements expected are comparable to those calculated in Appendix B, but the accelerations are considerably reduced because the allowed ductility reduces the maximum force the structure experiences.

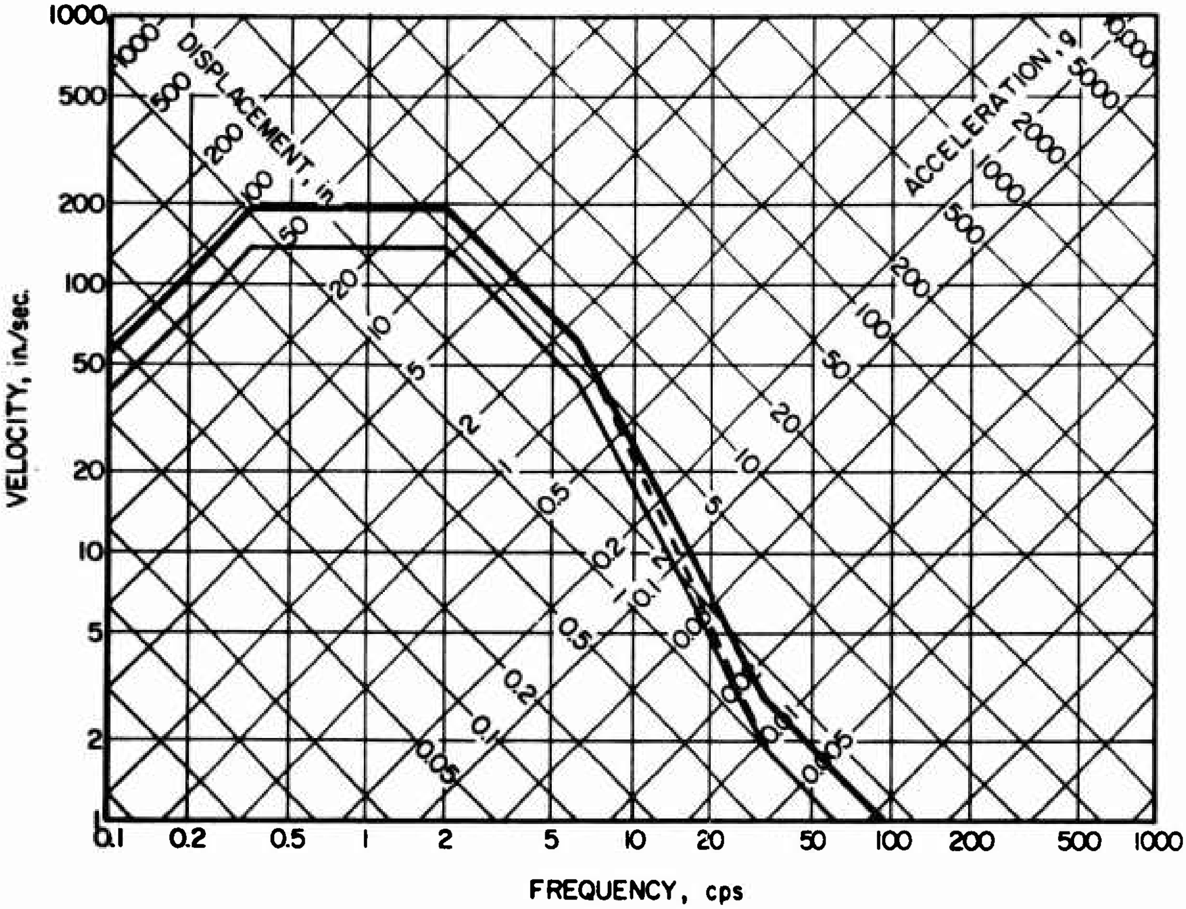

Figure C3. Design spectrum for elastoplastic system with $\mu = 1.5$.

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